On global solutions for reaction-diffusion systems with bounded total mass : old and new

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1) A 2×2 model system

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$$(MS)\begin{cases} \partial_t u - d_1 \Delta_x u = f(u,v) & \text{in } (0,T)\times\Omega \\ \partial_t v - d_2 \Delta_x v = g(u,v) & \text{in } (0,T)\times\Omega \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } (0,T)\times\partial\Omega \\ u(0,\cdot) = u_0(\cdot) \geq 0, \ v(0,\cdot) = & v_0(\cdot) \geq 0. \end{cases}$$

$d_1, d_2 > 0$, $f, g : [0,\infty)^2 \to \boldsymbol{R}$ **regular**,

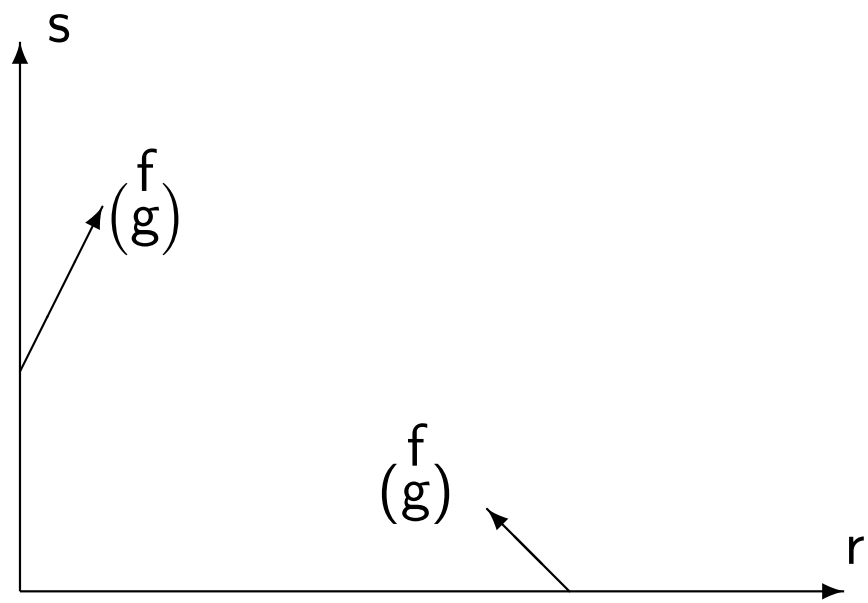
$\Omega \subset \boldsymbol{R}^N$, bounded open regular,

$u, v : (0,T)\times\Omega \to \boldsymbol{R}$ unknown functions.

We denote $Q_T = (0,T)\times\Omega$.

- **(P) Positivity preserving :** is equivalent to assuming f and g are quasi-positive which mean

$$\forall r, s \geq 0,\ f(0,s) \geq 0,\ g(r,0) \geq 0.$$



- $$(MS)\begin{cases} \partial_t u - d_1 \Delta u = f(u,v) & \text{on } Q_T \\ \partial_t v - d_2 \Delta v = g(u,v) & \text{on } Q_T \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } (0,T) \times \partial\Omega \\ u(0,\cdot) = u_0(\cdot) \geq 0, \ v(0,\cdot) = & v_0(\cdot) \geq 0. \end{cases}$$
- **(P) Positivity preserving:**
  $\forall r, s \geq 0, \ f(0,s) \geq 0, \ g(r,0) \geq 0.$
- **(M) 'Control' of mass :** $\forall r, s \geq 0,$

$$f(r,s) + g(r,s) \leq 0$$

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- **(P) Positivity preserving:**
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- **(M) 'Control' of mass :** $\forall r, s \geq 0,$
$$f(r,s) + g(r,s) \leq 0$$
- $\Rightarrow$ **a priori $L^1(\Omega)$-estimate, uniform in time:**
$$\forall t \geq 0,\quad \int_\Omega [u(t,x) + v(t,x)]dx \leq \int_\Omega [u_0(x) + v_0(x)]dx.$$
  Comes from integration on Ω:
$$\int_\Omega \partial_t(u(t)+v(t))dx - \int_{\partial\Omega}\left(d_1\frac{\partial u}{\partial \nu} + d_2\frac{\partial v}{\partial \nu}\right) = \int_\Omega (f(u,v)+g(u,v))dx \leq 0.$$

More generally

$$(\mathbf{M}') \quad af(r,s) + bg(r,s) \leq C[r+s+1]$$

implies control of the total mass:

$$\sup_{0\leq t\leq T<+\infty} \int_\Omega [u(t)+v(t)]dx < +\infty.$$

QUESTION:

Existence of global solutions in time under **(P)+(M)** *??*

or more generally under **(P)+ (M')** ??

2) Examples

2) Explicit examples

- **"Chemical morphogenetic process ("Brusselator")**

$$\begin{cases} \partial_t u - d_1 \Delta u = -uv^2 + b\, v \\ \partial_t v - d_2 \Delta v = uv^2 - (b+1)\, v + a \\ u_{|\partial\Omega} = b/a,\ v_{|\partial\Omega} = a, \\ a, b, d_1, d_2 > 0. \end{cases}$$

See also: Glycolosis model–Gray-Scott models

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See also: Glycolosis model–Gray-Scott models

- **Exothermic combustion in a gas**

$$\begin{cases} \partial_t Y - \mu \Delta Y = -H(Y,T) \\ \partial_t T - \lambda \Delta T = q\,H(Y,T), \end{cases}$$

$Y =$ concentration of a single reactant, $T =$ temperature, $\mu, \lambda, q > 0$. Here we have: $q\,f(r,s) + g(r,s) = 0$.
See also: JMcGough+R.Riley+H.G. Othmer models

Explicit examples 2

- **Lotka-Volterra Systems (Leung,...)**

$$\forall i = 1, ..., m, \quad \partial_t u_i - d_i \Delta u_i = -e_i u_i + \sum_{1 \leq j \leq m} p_{ij} u_i u_j,$$

where $e_i \in \boldsymbol{R}^+$, $p_{ij} \in \boldsymbol{R}$ such that,

$$\forall w \in \boldsymbol{R}^m, \ \sum_{i,j=1}^{m} a_i p_{ij} w_i w_j \leq 0 \text{ for some } a_i > 0.$$

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- **Model of diffusive calcium dynamics: H.G. Othmer**

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = \lambda(\gamma_0 + \gamma_1 u_4)(1 - u_1) - \frac{p_1 u_1^4}{p_2^4 + u_1^4} \\ \partial_t u_2 - d_2 \Delta u_2 = -k_1 u_2 + k_1' u_3 \\ \partial_t u_3 - d_3 \Delta u_3 = -k_1' u_3 - k_2 u_1 u_3 + k_1 u_2 + k_2' u_4 \\ \partial_t u_4 - d_4 \Delta u_4 = k_2 u_1 u_3 + k_3' u_5 - k_2' u_4 - k_3 u_1 u_4 \\ \partial_t u_5 - d_5 \Delta u_5 = k_3 u_1 u_4 - k_3' u_5. \end{cases}$$

Explicit examples 3

- **Modelization of pollutants transfer in atmospher ($N = 3$): R. Texier-Picard, M. Pierre, (degenerate diffusion)**:

$$\begin{cases} \partial_t \phi_i = d_i \, \partial^2_{zz} \phi_i + \omega \cdot \nabla \phi + f_i(\phi) + g_i, \ \forall i = 1...20, \\ + \text{ Bdy and initial conditions} \end{cases}$$

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- **The reaction terms:**

$$\begin{cases}
f_1(\phi) &= -k_1\phi_1 + k_{22}\phi_{19} + k_{25}\phi_{20} + k_{11}\phi_{13} + k_9\phi_{11}\phi_2 + k_3\phi_5\phi_2 \\
&+ k_2\phi_2\phi_4 - k_{23}\phi_1\phi_4 - k_{14}\phi_1\phi_6 + k_{12}\phi_{10}\phi_2 - k_{10}\phi_{11}\phi_1 - k_{24}\phi_{19}\phi_1, \\
f_2(\phi) &= k_1\phi_1 + k_{21}\phi_{19} - k_9\phi_{11}\phi_2 - k_3\phi_5\phi_2 - k_2\phi_2\phi_4 - k_{12}\phi_{10}\phi_2 \\
f_3(\phi) &= k_1\phi_1 + k_{17}\phi_4 + k_{19}\phi_{16} + k_{22}\phi_{19} - k_{15}\phi_3 \\
f_4(\phi) &= -k_{17}\phi_4 + k_{15}\phi_3 - k_{16}\phi_4 - k_2\phi_2\phi_4 - k_{23}\phi_1\phi_4 \\
f_5(\phi) &= 2k_4\phi_7 + k_7\phi_9 + k_{13}\phi_{14} + k_6\phi_7\phi_6 - k_3\phi_5\phi_2 + k_{20}\phi_{17}\phi_6 \\
f_6(\phi) &= 2k_{18}\phi_{16} - k_8\phi_9\phi_6 - k_6\phi_7\phi_6 + k_3\phi_5\phi_2 - k_{20}\phi_{17}\phi_6 - k_{14}\phi_1\phi_6 \\
f_7(\phi) &= -k_4\phi_7 - k_5\phi_7 + k_{13}\phi_{14} - k_6\phi_7\phi_6 \\
f_8(\phi) &= k_4\phi_7 + k_5\phi_7 + k_7\phi_9 + k_6\phi_7\phi_6 \\
f_9(\phi) &= -k_7\phi_9 - k_8\phi_9\phi_6 \\
f_{10}(\phi) &= k_7\phi_9 + k_9\phi_{11}\phi_2 - k_{12}\phi_{10}\phi_2 \\
f_{11}(\phi) &= k_{11}\phi_{13} - k_9\phi_{11}\phi_2 + k_8\phi_9\phi_6 - k_{10}\phi_{11}\phi_1 \\
f_{12}(\phi) &= k_9\phi_{11}\phi_2 \\
f_{13}(\phi) &= -k_{11}\phi_{13} + k_{10}\phi_{11}\phi_1 \\
f_{14}(\phi) &= -k_{13}\phi_{14} + k_{12}\phi_{10}\phi_2 \\
f_{15}(\phi) &= k_{14}\phi_1\phi_6 \\
f_{16}(\phi) &= -k_{19}\phi_{16} - k_{18}\phi_{16} + k_{16}\phi_4 \\
f_{17}(\phi) &= -k_{20}\phi_{17}\phi_6 \\
f_{18}(\phi) &= k_{20}\phi_{17}\phi_6 \\
f_{19}(\phi) &= -k_{21}\phi_{19} - k_{22}\phi_{19} + k_{25}\phi_{20} + k_{23}\phi_1\phi_4 - k_{24}\phi_{19}\phi_1 \\
f_{20}(\phi) &= -k_{25}\phi_{20} + k_{24}\phi_{19}\phi_1 .
\end{cases}$$

Explicit examples 4 : Reversible chemical reactions

- **Model of chemical reactions 1 (mass action law)**

$$A + B \rightleftharpoons C + D$$

$$\begin{cases} \partial_t a - d_1 \Delta a = -ab + cd \\ \partial_t b - d_2 \Delta b = -ab + cd \\ \partial_t c - d_3 \Delta c = ab - cd \\ \partial_t d - d_4 \Delta d = ab - cd. \end{cases}$$

Here $f_1 + f_2 + f_3 + f_4 = 0$.

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Here $f_1 + f_2 + f_3 + f_4 = 0$.

- **Model of chemical reactions 2 (mass action law)**

$$\alpha U + \beta V \rightleftharpoons \gamma W$$

$$\begin{cases} \partial_t u - d_1 \Delta u = \alpha(w^\gamma - u^\alpha v^\beta) \\ \partial_t v - d_2 \Delta v = \beta(w^\gamma - u^\alpha v^\beta) \\ \partial_t w - d_3 \Delta w = \gamma(-w^\gamma + u^\alpha v^\beta) \end{cases}$$

$\alpha, \beta, \gamma \geq 1$.
Here $\beta\gamma f_1 + \alpha\gamma f_2 + 2\alpha\beta f_3 = 0$.

3) CLASSICAL SOLUTIONS

3.1) The known results

Easy facts on the PDE system

$$(MS)\begin{cases} \partial_t u - d_1 \Delta u = f(u,v) & \text{in } Q_T \\ \partial_t v - d_2 \Delta v = g(u,v) & \text{in } Q_T \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega \\ u(0,\cdot) = u_0(\cdot) \geq 0, \ v(0,\cdot) = & v_0(\cdot) \geq 0. \end{cases}$$

- For $u_0, v_0 \in L^\infty(\Omega)$, local existence and uniqueness of classical solution (u,v) to (MS) are known.

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- More precisely, there exists $T > 0$ and a unique classical solution (u,v) of (MS) on $[0,T)\times\Omega$.

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- For $u_0, v_0 \in L^\infty(\Omega)$, local existence and uniqueness of classical solution (u,v) to (MS) are known.
- More precisely, there exists $T > 0$ and a unique classical solution (u,v) of (MS) on $[0,T)\times\Omega$.
- Moreover $u,\ v \geq 0$ thanks to **(P)**.

- The maximal time of existence T^* is characterized by

$$\sup_{t\in(0,T^*)} \left(\|u(t)\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)}\right) < +\infty \quad \Rightarrow \quad T^* = +\infty.$$

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- **A priori L^∞-estimates, uniform in time, imply global existence !**
- **But this type of estimates is far of being obvious for our system except in the case where diffusion coefficients are egal.**

Case : $d_1 = d_2 = d$

$$\partial_t[u+v] - d\Delta(u+v) = f(u,v) + g(u,v) \leq 0.$$

By maximum principle for the heat equation:

$$\forall t \in (0, T^*), \quad \|u(t) + v(t)\|_{L^\infty(\Omega)} \leq \|u_0 + v_0\|_{L^\infty(\Omega)}.$$

and, since $0 \leq u, v$, this implies L^∞-bounds on $u(t), v(t)$ and therefore

$$T^* = +\infty \text{ !!.}$$

Later on, we will assume $d_1 \neq d_2$.

A model example

- $$\begin{cases} \partial_t u - d_1 \Delta u = -uv^\beta \\ \partial_t v - d_2 \Delta v = uv^\beta \end{cases}$$

$\beta \geq 1$, + boundary conditions and initial data .

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- Theorem

$$\forall t \in (0, T^*), \ \|v(t)\|_{L^\infty(\Omega)} \leq C$$

and therefore $T^* = +\infty$.

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- Alikakos ($\beta < \dfrac{N+2}{N}$), Mazuda, Hollis-Martin-Pierre, ...

More generally

- $$\begin{cases} \partial_t u - d_1 \Delta u = f(u,v) \\ \partial_t v - d_2 \Delta v = g(u,v) \end{cases}$$

Theorem (Hollis-Martin-Pierre)

Assuming **(P)+(M)**, *if one of the components* u *or* v *is a priori uniformly bounded on* Q_{T^*}, *and if* f, g *grow at most like* **polynomials** *for* u, v *large, then the other component is also bounded and therefore* $T^* = +\infty$.

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- **Ingredient:** The L^p-estimate by duality introduced by Michel Pierre ($1 < p < +\infty$).

- **Theorem (Pierre's duality lemma)**

  Let $u, v \geq 0$ *such that*

$$\partial_t u - d_1 \Delta u + \partial_t v - d_2 \Delta v \leq 0.$$

  Then there exists $C = C(p, T, \Omega)$ *such that*

$$\forall 1 < p < +\infty, \ \|v\|_{L^p(Q_T)} \leq C(1 + \|u\|_{L^p(Q_T)}).$$

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  - **Open problem :**

$$(S) \begin{cases} \partial_t u - d_1 \Delta u = -u\, h(v) \\ \partial_t v - d_2 \Delta v = u\, h(v), \end{cases}$$

    when $h(v)(\geq 0)$ **grows faster than a polynomial as** $v \to +\infty$. For instance: $h(v) = e^{v^\beta}$ where $\beta > 1$.

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    when $h(v)(\geq 0)$ **grows faster than a polynomial as** $v \to +\infty$. For instance: $h(v) = e^{v^\beta}$ where $\beta > 1$.
  - **Curiously** , when

$$f(u, v) = -u^\alpha (1 + v^2) e^{v^2} \text{ and } g(u, v) = u^\alpha e^{v^2}.$$

    (S) has a global classical solution (**cf. Benachour-Rebiai, J.E.E, 2010**).

3.2) NEW RESULTS

Recall

$$\alpha U + \beta V \rightleftharpoons \gamma W$$

$$(S_{\alpha\beta\gamma}) \begin{cases} \partial_t u - d_1 \Delta u = \alpha(w^\gamma - u^\alpha v^\beta) \\ \partial_t v - d_2 \Delta v = \beta(w^\gamma - u^\alpha v^\beta) \\ \partial_t w - d_3 \Delta w = \gamma(-w^\gamma + u^\alpha v^\beta) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \\ u(0,.) = u_0 \geq 0 \ ; \ v(0,.) = v_0 \geq 0 \ ; \ w(0,.) = v_0 \geq 0 \end{cases}$$

$\alpha,\ \beta,\ \gamma \geq 1$

Previously, the following cases have been studied by several authors:

- **Case $\alpha = \beta = \gamma = 1$:**

  • In this case, global existence of classical solutions was first obtained by Rothe for dimensions $N \leq 5$.
  Later, global classical solutions have first been proved by Pierre for all dimensions N and then by Morgan, by Feng, by Kouachi, ...

  • Exponential decay towards equilibrium has been shown by Desvillettes-Fellner for bounded solutions.

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  • Exponential decay towards equilibrium has been shown by Desvillettes-Fellner for bounded solutions.
- **Case $\gamma = 1$ and arbitrary $\alpha, \beta \geq 1$:**
  In this case, global existence of classical solutions has been obtained by Feng in all dimensions N and more general boundary conditions.

3.2) NEW RESULTS

- **E.-H. L** proved global existence of classical solutions to System $(S_{\alpha\beta\gamma})$ in the following cases :
  - $\alpha + \beta < \gamma$;
  - $(d_1 = d_3$ or $d_2 = d_3)$ and for arbitrary $\alpha, \beta, \gamma \geq 1$;
  - $d_1 = d_2$ and for any (α, β, γ) such that $\alpha + \beta \neq \gamma$;
  - $1 < \gamma < \frac{N+6}{N+2}$ and for arbitrary $\alpha, \beta \geq 1$.

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- Recently Klemens Fellner and I have established global existence in the case where $|d_1 - d_3|$ or $|d_2 - d_3|$ is sufficiently small for any (α, β, γ).
- **As far as I know, the question of global existence in the case $1 < \gamma \leq \alpha + \beta$ is still OPEN**

Assume $|\Omega| = 1$.

- The System $(S_{\alpha\beta\gamma})$ with the conservation laws possesses a unique detailed balance equilibrium state, which are the unique positive constants $u_\infty, v_\infty, w_\infty$ balancing the three reversible reactions, i.e.

$$u_\infty^\alpha \, v_\infty^\beta = w_\infty^\gamma$$

and satisfying the conservation laws, i.e.

$$\gamma u_\infty + \alpha w_\infty = \int_\Omega \gamma u_0 + \alpha w_0, \qquad \gamma v_\infty + \beta w_\infty = \int_\Omega \gamma v_0 + \beta w_0.$$

Assume $|\Omega| = 1$.

- The System $(S_{\alpha\beta\gamma})$ with the conservation laws possesses a unique detailed balance equilibrium state, which are the unique positive constants $u_\infty, v_\infty, w_\infty$ balancing the three reversible reactions, i.e.

$$u_\infty^\alpha \, v_\infty^\beta = w_\infty^\gamma$$

and satisfying the conservation laws, i.e.

$$\gamma u_\infty + \alpha w_\infty = \int_\Omega \gamma u_0 + \alpha w_0, \qquad \gamma v_\infty + \beta w_\infty = \int_\Omega \gamma v_0 + \beta w_0.$$

- In fact, the uniqueness of the equilibrium $(u_\infty, v_\infty, w_\infty)$ follows from the uniqueness of the solution w_∞ of the transcendental equation

$$\left(\frac{M_1}{\gamma} - \frac{\alpha w_\infty}{\gamma}\right)^\alpha \left(\frac{M_2}{\gamma} - \frac{w_\infty \beta}{\gamma}\right)^\beta = w_\infty^\gamma, \qquad (1)$$

where the left-hand-side of (1) is strictly monotone decreasing in w_∞ while the right-hand-side of (1) is strictly monotone increasing in w_∞.

Finally, the System $(S_{\alpha\beta\gamma})$ features a Boltzmann-type entropy functional (or physically, a free energy functional)

$$E(u,v,w)(t) = \int_\Omega (u(\ln(u)-1) + v(\ln(v)-1) + w(\ln(w)-1))\,dx,$$

which dissipates according to the non-negative entropy dissipation functional $D = -\frac{d}{dt}E$.

Theorem (K. Fellner, E.-H.L, 2015)

Let (u_0, v_0, w_0) with finite mass and finite entropy $E(u_0, v_0, w_0)$. Then the solution (u, v, w) of System $(S_{\alpha\beta\gamma})$ satisfies the following exponential decay towards equilibrium :

$$\begin{aligned} &\|u(t)-u_\infty\|_{L^1(\Omega)} + \|v(t)-v_\infty\|_{L^1(\Omega)} + \|w(t)-w_\infty\|_{L^1(\Omega)} \\ &\leq C(E(u_0,v_0,w_0) - E(u_\infty,v_\infty,w_\infty))\,e^{-K\,t}, \qquad \text{a.e. } 0 \leq t < T, \end{aligned}$$

3.3) OPEN PROBLEMS 1

What happens when neither u nor v is a priori bounded?

• A typical example may be obtained as a perturbation of the previous model example:

$$\begin{cases} \partial_t u - d_1 \Delta u = \lambda u^p v^q - u^\alpha v^\beta \\ \partial_t v - d_2 \Delta v = -u^p v^q + u^\alpha v^\beta, \end{cases}$$

Note that : $f(u,v) + g(u,v) = (\lambda - 1) u^p v^q \leq 0$ if $\lambda \in [0,1]$.
But neither u or v is a priori bounded !

• Same situation for

$$\begin{cases} \partial_t u - d_1 \Delta u = -c(t,x) u^\alpha v^\beta \\ \partial_t v - d_2 \Delta v = c(t,x) u^\alpha v^\beta. \end{cases}$$

OK, if $c(x,t) \geq 0$. But **open problem** otherwise. Blow up may occur precisely in the region where the sign of $c(t,x)$ changes.

3.3) OPEN PROBLEMS 2

- **Model of chemical reactions 1 (mass action law)**

$$A + B \rightleftharpoons C + D$$

$$\begin{cases} \partial_t a - d_1 \Delta a = -ab + cd \\ \partial_t b - d_2 \Delta b = -ab + cd \\ \partial_t c - d_3 \Delta c = ab - cd \\ \partial_t d - d_4 \Delta d = ab - cd. \end{cases}$$

$N = 1, 2$ Goudon-Vasseur ; Prüss ; Caniso-Desvillettes-Fellner.

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$N = 1, 2$ Goudon-Vasseur ; Prüss ; Caniso-Desvillettes-Fellner.
OPEN PROBLEM for $N \geq 3$

- **Model of chemical reactions 2 (mass action law)**

$$\alpha U + \beta V \rightleftharpoons \gamma W$$

$$\begin{cases} \partial_t u - d_1 \Delta u = \alpha(w^\gamma - u^\alpha v^\beta) \\ \partial_t v - d_2 \Delta v = \beta(w^\gamma - u^\alpha v^\beta) \\ \partial_t w - d_3 \Delta w = \gamma(-w^\gamma + u^\alpha v^\beta) \end{cases}$$

OK for the case $\alpha + \beta < \gamma$, E.-H.L.
OPEN PROBLEM for $\alpha + \beta \geq \gamma > 1$

What about initial question ?

- **QUESTION :** Existence of global solutions in time

  under **(P)+(M)**??

  or more generally under **(P)+ (M')** ??

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  or more generally under **(P)+ (M')** ??

- Theorem (Michel PIERRE, Didier SCHMITT '97)

  One can find f, g, homogeneous polynomials, such that **(P)+(M)** *hold , but there exists $T^* < +\infty$ with*

$$\lim_{t \to T^*} \|u(t)\|_{L^\infty(\Omega)} = \lim_{t \to T^*} \|v\|_{L^\infty(\Omega)} = +\infty.$$

4) WEAK SOLUTIONS

- **NOW we will look for :**
  WEAK SOLUTIONS (which may go out of $L^\infty(\Omega)$ at some times)... but the nonlinearities $f(u,v), g(u,v)$ are in $L^1(Q_T)$ for all $T > 0$.
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- ...like $u(t,x) = \frac{1}{t^2+|x|^2}$ which is **weak** solution of $\partial_t u - \Delta u = c(t,x)\, u^2$ with $c(t,x)$ bounded ($N \geq 4$).

4.1) The known results

Recall the model system

$$(MS)\begin{cases} \partial_t u - d_1\Delta u = f(u,v) & \text{on } Q_T \\ \partial_t v - d_2\Delta v = g(u,v) & \text{on } Q_T \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } (0,T)\times\partial\Omega \\ u(0,\cdot) = u_0(\cdot) \geq 0,\ v(0,\cdot) = & v_0(\cdot) \geq 0. \end{cases}$$

Weak solution means : $f(u,v), g(u,v)$ are in $L^1(Q_T)$ for all $T > 0$ and (MS) is understood **in the sense of distributions** or, equivalently here, solution **in the sense of corresponding semigroups**

$$u(t) = S_{d_1}(t)u_0 + \int_0^t S_{d_1}(t-s)f(u,v)\,ds$$

$$v(t) = S_{d_2}(t)v_0 + \int_0^t S_{d_2}(t-s)g(u,v)\,ds$$

where $S_{d_i}(.)$ is the semigroup generated in $L^1(\Omega)$ by $-d_i\Delta$ with homogeneous Neumann boundary condition.

4.1) Known results : L^1-theorem for linear diffusion

$$(MS)\begin{cases} \partial_t u - d_1 \Delta u = f(u,v) & \text{on } Q_T \\ \partial_t v - d_2 \Delta v = g(u,v) & \text{on } Q_T \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } (0,T)\times\partial\Omega \\ u(0,\cdot) = u_0(\cdot) \geq 0,\ v(0,\cdot) = & v_0(\cdot) \geq 0. \end{cases}$$

Theorem (Michel Pierre, 2003, L^1-**Theorem**)

Assume **(P)+ (M)** *hold together with the a priori estimate*

$$\forall T > 0,\ \int_{Q_T} \left[|f(u,v)| + |g(u,v)|\right] dt\, dx \leq C(T) < +\infty.$$

Then (MS) *has a global weak solution for all* $(u_0, v_0) \in (L^1(\Omega)^+)^2$.

Application of L^1-theorem

- Recall that System

$$(S_{\exp})\begin{cases} \partial_t u - d_1\Delta u = -u^\alpha e^{v^\beta} & \text{in } Q_T \\ \partial_t v - d_2\Delta v = u^\alpha e^{v^\beta} & \text{in } Q_T \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega \\ u(0,\cdot) = u_0(\cdot) \geq 0,\ v(0,\cdot) = v_0(\cdot) \geq 0. \end{cases}$$

has **global classical solution** for any $\beta \leq 1$ and for nonnegative initial data $(u_0, v_0) \in L^\infty(\Omega)\times L^\infty(\Omega)$ and $\|u_0\|_{L^\infty(\Omega)} \leq \dfrac{8d_1d_2}{\alpha N(d_1-d_2)^2}$.

Haraux-Youkana for $\beta < 1$ and Barabanova for $\beta = 1$.

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- Thanks to L^1-**theorem**, System $(S_{\exp})$ has **global weak solution** for any $\beta > 0$ and for any nonnegative initial data $(u_0, v_0) \in L^1(\Omega)\times L^1(\Omega)$.

Estimate $\int_{Q_T} |f| + |g| \leq C$

$$(MS) \begin{cases} \partial_t u - d_1 \Delta u = f(u,v) & \text{in } Q_T \\ \partial_t v - d_2 \Delta v = g(u,v); & \text{in } Q_T \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega \\ u(0,\cdot) = u_0(\cdot) \geq 0, \ v(0,\cdot) = & v_0(\cdot) \geq 0. \end{cases}$$

Proposition (Michel Pierre)

Assume **(P)** *and*

$$(\mathbf{M})\ f + g \leq 0 \ \textbf{AND}\ (\mathbf{M}_\lambda)\ f + \lambda g \leq 0 \text{ for some } \lambda < 1.$$

Then, if (u,v) *is solution of* (MS) *on* $(0,T)$,

$$\int_{Q_T} [|f(u,v))| + |g(u,v)|]\, dt\, dx \leq M = M(data) < +\infty.$$

That is to say : the nonlinearities $f(u,v), g(u,v)$ are a priori bounded in $L^1(Q_T)$ for all $T > 0$.

Proof of the L^1-a priori estimate $\int_{Q_T} |f| + |g| \leq C$

For $\mu = 1$ and $\mu = \lambda$, we have

$$\partial_t \int_\Omega (u + \mu v) + \int_\Omega -[f(u,v) + \mu g(u,v)] = 0.$$

But $-[f(u,v) + \mu g(u,v)] \geq 0$. After integration in time on $(0, T)$

$$\int_\Omega [u(T) + \mu v(T)] + \int_{Q_T} |f(u,v) + \mu g(u,v)| \leq \int_\Omega u_0 + \mu v_0.$$

It follows that, for $\mu = 1$ and for $\mu = \lambda \neq 1$

$$\|f(u,v) + \mu g(u,v)\|_{L^1(Q_T)} \leq C,$$

and therefore

$$\|f(u,v)\|_{L^1(Q_T)} \leq C, \text{ and } \|g(u,v)\|_{L^1(Q_T)} \leq C.$$

More generally

- **L^1-theorem holds for systems with $\ell \geq 2$ equations.**

$$(RDSLD)\begin{cases} 1 \leq i \leq \ell & & & \\ \partial_t u_i - d_i \Delta u_i & = & f_i(u_1, \cdots, u_\ell) & (0,+\infty) \times \Omega, \\ \frac{\partial u_i}{\partial \nu} & = & 0 & (0,+\infty) \times \partial\Omega \\ u_i(0,x) & = & u_{0,i}(x) \geq 0 & x \in \Omega, \end{cases}$$

where $d_i \in (0,+\infty)$.

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where $d_i \in (0,+\infty)$.

- Theorem (M. Pierre, 2010)

Assume $f = (f_1, \cdots, f_\ell)$ satisfies **(P)**, **(M)** *and assume there is an L^1-a priori estimate on the nonlinearities*

$$\sup_{1\leq i\leq \ell} \|f_i(u)\|_{L^1(Q_T)} \leq C(T) \text{ for all } T > 0.$$

Then, there exists a global **weak solution** *to $(RDSLD)$ for all $u_0 = (u_{0,1}, \cdots, u_{0,\ell}) \in (L^1(\Omega)^+)^\ell$.*

Step 1 of the proof L^1-compactness

- Truncating the f_i and the $u_{0,i}$. Then the system

$$\begin{cases} 1 \leq i \leq \ell & & & \\ \partial_t(u_{i,n}) - d_i \Delta u_{i,n} & = & f_{i,n}(u_{1,n}, \cdots, u_{\ell,n}) & (t,x) \in (0,+\infty) \times \Omega \\ u_{i,n}(t,x) & = & 0 & (t,x) \in (0,+\infty) \times \partial \\ u_{i,n}(0,x) & = & u_{0,i,n}(x) & x \in \Omega, \end{cases}$$

has global solution $u_n = (u_{1,n}, \cdots, u_{\ell,n})$ and $\|f_{i,n}(u_n)\|_{L^1(Q_T)}$ **bounded independtly of** n.

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- **Lemma (Baras-Hassan-Véron)** : The mapping $(w_0, H) \in L^1(\Omega) \times L^1(Q_T) \mapsto w \in L^1(Q_T)$ where

$$\partial_t w - d\Delta w = H \text{ in } Q_T, \ w = 0 \text{ on } (0,T) \times \partial\Omega, \ w(0,.) = w_0 \text{ in } \Omega.$$

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- This lemma holds for the nonlinear porous media eqution (Baras).

$$\partial_t w - d\Delta w^q = H \text{ in } Q_T, \ w = 0 \text{ on } (0,T) \times \partial\Omega, \ w(0,.) = w_0 \text{ in } \Omega.$$

Step 2 : consequences of L^1-compactness

- $1 \leq i \leq \ell$; $\partial_t(u_{i,n}) - d_i \Delta u_{i,n} = f_{i,n}(u_n)$ and $\int_{Q_T} |f_{i,n}(u_n)| \leq C(T)$.

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- Since $\{f_{i,n}(u_n)\}$ is bounded in $L^1(Q_T)$ and $\{u_{0,i,n}\}$ is bounded in $L^1(\Omega)$, $\{u_{i,n}\}$ is relatively compact in $L^1(Q_T)$.

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- Since $\{f_{i,n}(u_n)\}$ is bounded in $L^1(Q_T)$ and $\{u_{0,i,n}\}$ is bounded in $L^1(\Omega)$, $\{u_{i,n}\}$ is relatively compact in $L^1(Q_T)$.
- Up to a subsequence, we may assume : $\forall 1 \leq i \leq \ell$
  $u_{i,n} \to u_i$ in $L^1(Q_T)$, $\forall T > 0$ and a.e Q_∞
  $f_{i,n}(u_n) \to f_i(u)$ a.e Q_∞ .
  Recall $f_{i,n}(u_n) = \dfrac{f_i(u_n)}{1 + \frac{1}{n}\sum_j |f_j(u_n)|}$
- QUESTION : can we claim that $\partial_t u_i - d_i \Delta u_i = f_i(u)$?
  Note that in general $f_{i,n}(u) \to f_i(u) + \mu_i$, where $\mu_i =$ measure.

Step 3, if the limit is a supersolution, then is finished (1)

- If we moreover knew that $\{f_{i,n}(u_n)\}$ is **uniformly integrable**, we could conclude $\mu_i = 0$ by Vitali's theorem. But this is valid only in some cases.

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$$\partial_t u_i - d_i \Delta u_i = f_i(u) + \mu_i \text{ where } \mu_i = \text{nonnegative measure.}$$

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- Then, we easily deduce that u is a **solution** thanks to the structure $\sum_i f_i(u) \leq 0$ and Fatou's Lemma.
- In fact, for any test-functions $\phi \in \mathcal{C}_0^\infty(Q_T)$, $\phi \geq 0$

$$-\int_{Q_T} \left(\partial_t \phi (\sum_i u_{i,n}) + \Delta \phi (\sum_i d_i u_{i,n}) \right) = \int_{Q_T} \phi \sum_i f_{i,n}(u_n).$$

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- Recall $f_{i,n}(u_n) \to f_i(u)$ *a.e* and $-\sum_i f_{i,n}(u_n) \geq 0$.

Step 3, if the limit is a supersolution, then is finished (2)

- By letting $n \to +\infty$, Fatou's lemma implies,

$$-\int_{Q_T}\left(\partial_t\phi(\sum_i u_i)+\Delta\phi(\sum_i d_i u_i)\right)=\int_{Q_T}\phi\sum_i f_i(u).$$

whence

$$\int_{Q_T}\phi\sum_i(f_i(u)+\mu_i)\leq\int_{Q_T}\phi\sum_i f_i(u)$$

which implies
$\sum_i \mu_i \leq 0$ and therefore : $\forall 1 \leq i \leq \ell,\ \mu_i = 0.$
Then for all i, $\mu_i = 0$ and $\partial_t u_i - d_i\Delta u_i = f_i(u)$.

Step 4 : How to prove that the limit is a supersolution (1)

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$$\partial_t u_n - d\Delta u_n = f_n(u_n) \text{ with } \int_{Q_T} |f_n(u_n)| \leq C.$$

- For a regularized truncation T_k, concave on $[0, +\infty)$, we have for all $k \geq 1$

$$\partial_t Tk(u_n) - d\Delta Tk(u_n) = T_k'(u_n) f_n(u_n) - T_k''(u_n)|\nabla u_n|^2 \geq T_k'(u_n) f_n($$

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- For k fixed, we may pass to the limit as $n \to +\infty$.
  $T_k'(u_n) f_n(u_n) \to T_k'(u) f(u)$ $a.e$ and is bounded independently of n. Thus the convergence holds in $L^1(Q_T)$ and

$$\partial_t Tk(u) - d\Delta Tk(u) \geq T_k'(u) f(u).$$

- Finally $k \to +\infty$ leads to ($f(u) \in L^1(Q_T)$ by Fatou's Lemma) $\partial_t u - d\Delta u \geq f(u)$.

Step 4 : How to prove that the limit is a supersolution (2)

- For a system ($\ell > 1$), the method does not work directly since

$$\partial_t Tk(u_{i,n}) - d_i \Delta Tk(u_{i,n}) \geq T_k'(u_{i,n}) f_{i,n}(u_n),$$

and even for k fixed, one cannot control $f_{i,n}(u_{1,n}, \cdots, u_{\ell,n})$.

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and even for k fixed, one cannot control $f_{i,n}(u_{1,n}, \cdots, u_{\ell,n})$.

- **IDEA** : use $w_{i,n} := T_k(u_{i,n} + \eta \sum_{j \neq i} u_{j,n})$ with $\eta > 0$. Then

$w_{i,n}$ satisfies

$$\partial_t w_{i,n} - d_i \Delta w_{i,n} \geq T_k'(w_{i,n}) \left(f_{i,n}(u_n) + \eta \sum_{j \neq i} f_{j,n}(u_n) + S_{i,n}(\eta, k) \right),$$

where $S_{i,n}(\eta, k) = \eta \sum_{j \neq i} (d_j - d_i) \Delta u_{j,n}$.

Step 4 : How to prove that the limit is a supersolution (2)

- For a system ($\ell > 1$), the method does not work directly since

$$\partial_t Tk(u_{i,n}) - d_i \Delta Tk(u_{i,n}) \geq T_k'(u_{i,n}) f_{i,n}(u_n),$$

and even for k fixed, one cannot control $f_{i,n}(u_{1,n}, \cdots, u_{\ell,n})$.

- **IDEA** : use $w_{i,n} := T_k(u_{i,n} + \eta \sum_{j \neq i} u_{j,n})$ with $\eta > 0$. Then

$w_{i,n}$ satisfies

$$\partial_t w_{i,n} - d_i \Delta w_{i,n} \geq T_k'(w_{i,n}) \left(f_{i,n}(u_n) + \eta \sum_{j \neq i} f_{j,n}(u_n) + S_{i,n}(\eta, k) \right),$$

where $S_{i,n}(\eta, k) = \eta \sum_{j \neq i} (d_j - d_i) \Delta u_{j,n}$.

- For k, η fixed, we can pass to the limit as $n \to +\infty$ in the nonlinear term $T_k'(w_{i,n}) \left(f_{i,n}(u_n) + \eta \sum_{j \neq i} f_{j,n}(u_n) \right)$

Step 4 : How to prove that the limit is a supersolution (4)

- But we need to control the corrective term

$$R_i(k, \eta, n) := T_k'(w_{i,n}) S_{i,n}(\eta, k).$$

Step 4 : How to prove that the limit is a supersolution (4)

- But we need to control the corrective term

$$R_i(k,\eta,n) := T_k'(w_{i,n})S_{i,n}(\eta,k).$$

- To control the corrective term, we prove that for all $\phi \in \mathcal{C}_0^\infty(Q_T)$, $\phi \geq 0$

$$(*) \qquad \left| \int_{Q_T} R_i(k,\eta,n)\phi \right| \leq C(k,\phi)\eta^{1/2}.$$

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$$(*) \qquad \left| \int_{Q_T} R_i(k,\eta,n)\phi \right| \leq C(k,\phi)\eta^{1/2}.$$

- By letting successively $n \to +\infty$, $\eta \to 0$, $k \to +\infty$, we prove that the limit u_i is a **supersolution** *i.e*

$$\partial_t u_i - d_i \Delta u_i \geq f_i(u_1, \cdots, u_\ell) \text{ against test functions } \geq 0.$$

Step 4 : How to prove that the limit is a supersolution (5)

- $(*)$ is a consequence of an interesting L^1-estimate. Let

  $\partial_t w - d\Delta w = H$ in Q_T, $w = 0$ on $(0,T)\times\partial\Omega$, $w(0,.) = w_0$ in Ω.

  Then $\displaystyle\int_{[|w|\le k]} |\nabla w|^2 \le k\left(\|w_0\|_{L^1(\Omega)} + \|H\|_{L^1(Q_T)}\right).$

  • **Main estimate** for $\eta \to 0$: $\displaystyle\int_{[u_{i,n}\le k]} |\nabla u_{i,n}|^2 \le Ck.$

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  • **Main estimate** for $\eta \to 0$: $\displaystyle\int_{[u_{i,n}\le k]} |\nabla u_{i,n}|^2 \le Ck.$

- **Proof** : Multiply the equation in w by $T_k(w)$!

4.2) Known results : L^2-theorem for linear diffusion (1)

$$(RDSLD)\begin{cases} 1 \le i \le \ell & & & \\ \partial_t u_i - d_i \Delta u_i & = & f_i(u_1, \cdots, u_\ell) & (t,x) \in (0,+\infty) \times \Omega, \\ u_i(t,x) & = & 0 & (t,x) \in (0,+\infty) \times \partial\Omega \\ u_i(0,x) & = & u_{0,i}(x) & x \in \Omega, \end{cases}$$

where $d_i \in (0,+\infty)$

Theorem (Michel Pierre, L^2-**theorem**)

Assume **(P)+(M)** *and*

$$\forall 1 \le i \le \ell, \forall r \in [0,+\infty)^\ell, |f_i(r)| \le C(1 + \sum_{1 \le j \le \ell} r_j^2).$$

then , System $(RDSLD)$ *has a global weak solution for any nonnegative initial data* $u_0 \in (L^2(\Omega)^+)^\ell$.

Proof of L^2-theorem for **linear diffusion**

- Lemma (Michel Pierre and Didier Schmitt, 90', L^2-**Estimate**)
  Assume $f = (f_1, \cdots, f_\ell)$ *satisfies* **(P)** *and* **(M)**. *Then, the following a priori estimate holds for solutions of* $(RDSLD)$ *:*

$$\forall 1 \leq i \leq \ell, \forall T > 0, \|u_i\|_{L^2(Q_T)} \leq C = C\left(T, d_i, \sum_i \|u_{0,i}\|_{L^2(\Omega)}\right).$$

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  - **Proof :** By L^2-theorem + at most quadratic growth of f_i, we have the a priori estimate for the solutions of System $(RDSLD)$

$$\forall 1 \leq i \leq \ell, \int_{Q_T} |f_i(u)| dt dx \leq C(T).$$

  Then , we apply L^1-theorem.

4.2) Known results for linear diffusion (2)

Applications of L^2-theorem to quadratic systems with linear diffusion

- Applies to the reversible chemical reaction :
  $U_1 + U_3 \rightleftharpoons U_2 + U_4$.

$$\text{for } 1 \leq i \leq 4,\ \partial_t u_i - d_i \Delta u_i = (-1)^i (u_1 u_3 - u_2 u_4).$$

L. Desvillettes-K. Fellner-M. Pierre- J. Vovelle.

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- Applies to Lotka-Volterra Systems :

$$\forall i = 1, ..., m,\ \ \partial_t u_i - d_i \Delta u_i = -e_i u_i + \sum_{1 \leq j \leq m} p_{ij} u_i u_j,$$

where $e_i \in \boldsymbol{R}^+$, $p_{ij} \in \boldsymbol{R}$ such that,

$$\forall w \in \boldsymbol{R}^m,\ \sum_{i,j=1}^{m} a_i p_{ij} w_i w_j \leq 0 \text{ for some } a_i > 0.$$

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- and many others ...

Improvements of L^2-estimates

$L^{2+\varepsilon}$-estimate by J.A. Canizo, L. Desvillettes and K. Fellner

Lemma (J.A. Canizo, L. Desvillettes and K. Fellner, 2012)

Assume $f = (f_1, \cdots, f_m)$ *satisfies* **(P)** *and* **(M)**. *Then, the following a priori estimate holds for solutions of* $(RDSLD)$ *: there exists* $\varepsilon = \varepsilon(N)$ *such that :*

$$\forall 1 \leq i \leq m, \forall T > 0, \|u_i\|_{L^{2+\varepsilon}(Q_T)} \leq C$$

where $C = C\left(T, d_i, \sum_i \|u_{0,i}\|_{L^{2+\varepsilon}(\Omega)}\right)$.

4.3) New results : Nonlinear Degenerate Diffusion (1)

- **Question : What about systems with Nonlinear Degenerate Diffusion of porous media type $\Delta u_i^{m_i}$ with homogenous boundary Dirichlet condition and $m_i \in [1, +\infty)$?**

$$(RDSNLD) \begin{cases} 1 \leq i \leq \ell & & & \\ \partial_t u_i - d_i \Delta(u_i^{m_i}) & = & f_i(u_1, \cdots, u_\ell) & (0, +\infty) \times \Omega, \\ u_i(t, x) & = & 0 & (0, +\infty) \times \partial\Omega \\ u_i(0, x) & = & u_{0,i}(x) \geq 0 & x \in \Omega, \end{cases}$$

where $d_i \in (0, +\infty)$.

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where $d_i \in (0, +\infty)$.

- **We use the same ideas as linear diffusion.**

• Truncating the f_i and the $u_{0,i}$, to obtain global solutions $u_n = (u_{1,n}, \cdots, u_{\ell,n})$.

• Compactness of the mapping $(w_0, H) \in L^1(\Omega) \times L^1(Q_T) \mapsto w \in L^1(Q_T)$ where

$$\partial_t w - d\Delta w^q = H \text{ in } Q_T, \ w = 0 \text{ on } Q_T, \ w(0, .) = w_0 \text{ in } \Omega.$$

4.3) New results : Nonlinear degenerate diffusion (2)

However, this not sufficient to pass to the limit for the nonlinear terms $(u_{i,n})^{m_i}$!!

For this, we will use the more precise estimates on $(u_{i,n})^{m_i}$ and their gradients, namely:

Lemma (Lukkari (2010), Abdellaoui-Peral-Walias (2012))

Let w *be the solution of*

$$\partial_t w - d\Delta w^q = H \text{ in } Q_T,\ w = 0 \text{ on } Q_T,\ w(0,.) = w_0 \text{ in } \Omega.$$

Then,

$$\int_{Q_T} |w|^{q\alpha} \leq C_1 \ \text{ for } \ 0 < \alpha < 1 + \frac{2}{qN}, \tag{2}$$

$$\int_{Q_T} |\nabla w^q|^{\beta} \leq C_2 \ \text{ for } \ 1 \leq \beta < 1 + \frac{1}{1+qN}, \tag{3}$$

where $C_i = C_i\left(T, \alpha, \beta, q, \|w_0\|_{L^1(\Omega)}, \|H\|_{L^1(Q_T)}\right)$.

4.3)New results : Nonlinear degenerate diffusion (3)

- Use of the equation satisfied by

$$w_{i,n,\eta,k} = \partial_t \left(T_k(u_{i,n}) T_k'(\eta \sum_{j \neq i} (v_{j,n})^{m_j}) \right)$$

$$- \operatorname{div} \left(T_k'(u_{i,n}) T_k' \left(\eta \sum_{j \neq i} (v_{j,n})^{m_j} \right) \nabla (u_{i,n})^{m_i} \right).$$

This approximation is more adapted to the case of nonlinear diffusion.

- The limit u_i is a **supersolution** by letting successively :

$$n \to +\infty, \ \eta \to 0, \ k \to +\infty.$$

- Main estimates for $\eta \to 0$:

$$\int_{[u_{i,n} \leq k]} |\nabla (u_{i,n})^{m_i}|^2 \leq C k^{m_i} \text{ and } \int_{[u_{i,n} \leq k]} |\nabla u_{i,n}|^2 \leq C k^{2-m_i}.$$

where $C = C(\|u_{0,i}\|_{L^1(\Omega)}, \|f_i\|_{L^1(Q_T)})$.

Nonlinear degenerate diffusion (4)

• To go from a supersolution to a solution, we argue as in the linear diffusion.

Whence

Theorem (M. Pierre, E-H.L, 2015)

Assume $f = (f_1, \cdots, f_\ell)$ *satisfies* **(P)**, **(M)** *and assume there is an* L^1- *a priori estimate on the nonlinearities*

$$\sup_{1\leq i\leq m} \|f_i(u)\|_{L^1(Q_T)} \leq C(T) \text{ for all } T > 0.$$

Then, for $m_i < 2$, *there exists a global* **weak solution** *to* (*RDSNLD*) *for all* $u_0 = (u_{0,1}, \cdots, u_{0,\ell}) \in (L^1(\Omega)^+)^\ell$.

Another approach : Extention of L^2-estimates (4)

New results in nonlinear diffusion

$$(RDSNLD)\begin{cases} 1 \le i \le \ell & & & \\ \partial_t u_i - \Delta u_i^{m_i} & = & f_i(u_1, \cdots, u_\ell) & (t,x) \in (0,+\infty) \times \Omega, \\ u_i(t,x) & = & 0 & (t,x) \in (0,+\infty) \times \partial\Omega \\ u_i(0,x) & = & u_{0,i}(x) \ge 0 & x \in \Omega, \end{cases}$$

where $m_i \ge 1$.

Proposition (M. Pierre, E-H.L, 2014)

Assume $f = (f_1, \cdots, f_\ell)$ satisfies **(P)** *and* **(M)**. *Then , the following a priori estimate holds for solutions of $(RDSNLD)$:*
$\forall 1 \le i \le m, \forall T > 0$

$$\|u_i\|_{L^{m_i+1}(Q_T)} \le C = C\left(T, \sum_i \|u_{0,i}\|_{L^2(\Omega)}\right). \qquad (4)$$

Another approach : Extention of L^2-estimates (5)

Proof.

For the clarity, we treat the case $\ell = 2$. As usual we make the computations for regular enough solutions to (S_2) and they are preserved by passage to the limit.

- Thanks to relation $f_1 + f_2 \leq 0$, we have

$$\partial_t(u_1 + u_2) - \Delta(u_1^{m_1} + u_2^{m_2}) = f_1(u_1, u_2) + f_2(u_1, u_2) \leq 0. \quad (5)$$

- We now integrate (5) in t to obtain

$$(u_1 + u_2)(t) - \Delta\left(\int_0^t u_1^{m_1} + u_2^{m_2}\right) \leq u_{0,1} + u_{0,2}. \quad (6)$$

Another approach : Extention of L^2-estimates (6)

Let us denote $Z(t,x) := \int_0^t (u_1^{m_1}(s,x) + u_2^{m_2}(s,x))ds$. Since $(u_1(t) + u_2(t)) \geq 0$, (6) implies

$$-\Delta Z(t,x) \leq u_{0,1} + u_{0,2}.$$

Moreover, $Z(t) = 0$ on $\partial\Omega$ and $Z(t) \geq 0$ in Ω, then

$$\|Z(t)\|_{L^\infty(\Omega)} \leq C\|u_{0,1} + u_{0,2}\|_{L^\infty(\Omega)}.$$

Multiplying (6) by $u_1^{m_1}(t) + u_2^{m_2}(t) = \partial_t Z$ and integrating on Ω to obtain

$$\int_{Q_T} (u_1+u_2)(u_1^{m_1}+u_2^{m_2}) + \frac{1}{2}\int_\Omega |\nabla Z(T)|^2 \leq \int_\Omega (u_{0,1}+u_{0,2})Z(T) \leq C.$$

which gives (4).

Another approach : Extention of L^2-estimates (7)

Corollary (M. Pierre, E-H.L, 2014)

Assume $f = (f_1, \cdots, f_\ell)$ *satisfies* **(P)**,**(M)** *and*

$$|f_i(u)| \leq C \left(\sum_j (u_j)^{m_j+1-\varepsilon} + 1 \right). \tag{7}$$

Then, there exists a global **weak solution** *to* $(SRNLD)$ *for all* $u_0 = (u_{0,1}, \cdots, u_{0,\ell}) \in (L^2(\Omega)^+)^\ell$.

Another approach : Extention of L^2-estimates (8)

Remark. When $\min_{1\leq i\leq \ell} m_i > 1$, condition (7) is satisfied for any non linearity with a growth at most quadratic at infinty as can be seen much in chemistry and biochemistry models. It is of course all the more general as the m_i are big. It should be noticed that there is no restriction on m_i and N.

Idea of the proof :
$\{f_{i,n}(u_{i,n}, \cdots, u_{\ell,n})\}$ is bounded in $L^1(Q_T)$ indepently of n and even **uniformly integrable**.

Remark. All other structure which ensures uniform integrability gives global exitence of weak solutions to $(SRNLD)$.

Another approach : Extention of L^2-estimates (9)

Example : linear diffusion

$$U_1 + U_3 \rightleftharpoons U_2 + U_4$$

$$(LD)\begin{cases} \partial_t u_1 - d_1\Delta u_1 = u_2u_4 - u_1u_3 & (t,x)\in]0,+\infty[\times\Omega, \\ \partial_t u_2 - d_2\Delta u_2 = -u_2u_4 + u_1u_3 & (t,x)\in]0,+\infty[\times\Omega, \\ \partial_t u_3 - d_3\Delta u_3 = u_2u_4 - u_1u_3 & (t,x)\in]0,+\infty[\times\Omega, \\ \partial_t u_4 - d_4\Delta u_4 = -u_2u_4 + u_1u_3 & (t,x)\in]0,+\infty[\times\Omega, \\ u_i(0,x) = u_{0,i}(x) & x\in\Omega, \\ u_i(t,x) = 0 & (t,x)\in(0,+\infty)\times\partial\Omega \end{cases}$$

$\rightarrow$ Global existence of classical solution for $N = 1$ by L. Desvillettes-K. Fellner ;

$\rightarrow$ Global existence of weak solution for all N by L. Desvillettes-K. Fellner-M. Pierre-J. Vovelle.

$\rightarrow$ Global existence of classical solution for $N \leq 2$ by Y. Pruss, Goudon-Vasseur, J.A Caniso-L. Desvillettes-K. Fellner ;

$\rightarrow$ Open for classical solutions when $N \geq 3$

Another approach : Extention of L^2-estimates (10)

Example : nonlinear diffusion

$$(NLD)\begin{cases} \partial_t u_1 - d_1\Delta(u_1)^{m_1} = u_2u_4 - u_1u_3 & (t,x)\in]0,+\infty[\times\Omega, \\ \partial_t u_2 - d_2\Delta(u_2)^{m_2} = u_2u_4 - u_1u_3 & (t,x)\in]0,+\infty[\times\Omega, \\ \partial_t u_3 - d_3\Delta(u_3)^{m_3} = -u_2u_4 + u_1u_3 & (t,x)\in]0,+\infty[\times\Omega, \\ \partial_t u_4 - d_4\Delta(u_4)^{m_4} = -u_2u_4 + u_1u_3 & (t,x)\in]0,+\infty[\times\Omega, \\ u_i(0,x) = u_{0,i}(x)\geq 0 & x\in\Omega, \\ u_i(t,x) = 0 & (t,x)\in(0,+\infty)\times\partial\Omega \end{cases}$$

The system (NLD) has a global weak solution :

- if $\min_{1\leq i\leq 4} m_i > 1$, we apply the corollary ;
- iff $\min_{1\leq i\leq 4} m_i = 1$, entropy inequality implies uniform integrablity.

4.3) OPEN PROBLEMS

1) Going back to the initial system

$$(S_2)\begin{cases} 1 \leq i \leq \ell & & & \\ \partial_t u_i - \Delta u_i^{m_i} & = & f_i(u_1, \cdots, u_m) & (t,x) \in (0,+\infty) \times \Omega, \\ u_i(t,x) & = & 0 & (t,x) \in (0,+\infty) \times \partial\Omega \\ u_i(0,x) & = & u_{0,i}(x) \geq 0 & x \in \Omega, \end{cases}$$

What's happen if $\max\limits_{1 \leq i \leq \ell} m_i \geq 2$?

2) If we have $\Delta_{p_i}(u_i) = \mathrm{div}(|\nabla u_i|^{p_i - 2} |\nabla u_i|)$ instead $\Delta(u_i)^{m_i}$?

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