Asymptotic analysis of some PDE's

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2ème Workshop Algéro-français EDP et Applications, 2017, TLEMCEN

Outline

Let Ω be a bounded open set of $\mathbb{R}^n$, p be a real number, with $1 < p < \infty$ and q its conjugate. We denote by $\mathcal{K}$ a closed convex subset of $W_0^{1,p}(\Omega)$ containing 0 and satisfying

$$\max(u,v)\,,\ \min(u,v) \in \mathcal{K},\ \ \forall u,v \in \mathcal{K}. \tag{1}$$

For example:
- *equations;* $\mathcal{K} = W_0^{1,p}(\Omega)\,,$
- *obstacle problems;*
$\mathcal{K} = \left\{u \in W_0^{1,p}(\Omega) : u(x) \geq \psi(x)\,,\ \text{for a.e. } x \in \Omega_0\right\}$, Ω_0 is a subset of Ω and ψ is a given function on Ω_0,
- *elasto-plastic torsion problem;*
$\mathcal{K} = \left\{u \in W_0^{1,p}(\Omega) : |\nabla u(x)| \leq c,\ \text{for a.e. } x \in \Omega_0\right\}$, $c \geq 0$.

Now, let $a(x,\xi) = (a_i(x,\xi))_{1 \leq i \leq n}$ and $a_0(x,\xi)$ be a family of Carathéodory functions defined on $\Omega \times \mathbb{R}^{n+1}$ and satisfying for all $\xi = (\xi_i)_i \,, \xi' = \left(\xi'_i\right)_i \in \mathbb{R}^{n+1}$ and for a.e. x in Ω, there exist nonnegative constants $\alpha, \ \beta$, $\vartheta \in L^q(\Omega)$ such that

$$\sum_{0\leq i\leq n} a_i(x,\xi)\xi_i \geq \alpha \sum_{1\leq i\leq n} |\xi_i|^p\,, \tag{2}$$

$$\sum_{0\leq i\leq n} \left(a_i(x,\xi) - a_i(x,\xi')\right)\left(\xi_i - \xi_i'\right) \geq 0, \tag{3}$$

$$|a_i(x,\xi_0,\xi_1,\ldots,\xi_n)| \leq \vartheta(x) + \beta \sum_{0\leq i\leq n} |\xi_i|^{p-1}\,. \tag{4}$$

Then for f in $L^q(\Omega)$, we consider u solution of the following nonlinear variational inequality

$$\left|\begin{array}{l} u \in \mathcal{K}, \\ \langle Au, v-u\rangle_\Omega \geq \displaystyle\int_\Omega f\,(v-u)\,dx, \ \ \forall v \in \mathcal{K}, \end{array}\right. \tag{5}$$

where A is a nonlinear operator defined from $W_0^{1,p}(\Omega)$ into its dual by

$$\langle Au, v\rangle = \int_\Omega a(x,u,\nabla u)\cdot\nabla v dx + \int_\Omega a_0(x,u,\nabla u) v dx,\ \forall v \in W_0^{1,p}(\Omega)\,.$$

Example

For $n = 1$ and $p = 2$, let $v \in H_0^1(0,1)$ be the nonnegative function defined by

$$v(x) = \frac{3\sqrt{3}}{2} x \chi_{(0,\frac{1}{3})} + \sin(\pi x)\, \chi_{(\frac{1}{3},1)},$$

where χ_A denotes the characteristic function of the set A. Consider

$$\mathcal{K} = \left\{ w \in H_0^1(0,1) : w \geq v \text{ a.e. in } (0,1) \right\},$$

$a_0 = 0$ and $a : \mathbb{R} \to \mathbb{R}$ is a single-valued function (Graph).
A is monotone and satisfies the above coercivness and growth conditions.
Hence, the solution to (5), for

$$f(x) = \pi^2 \sin(\pi x)\, \chi_{(\frac{1}{3},1)}.$$

Example

Hence, the solution to (5), for

$$f(x) = \pi^2 \sin(\pi x)\, \chi_{\left(\frac{1}{3},1\right)}.$$

exists and moreover it is not necessary unique, it is enough to check that the functions

$$u_\lambda = \lambda \sin(\pi x) + (1-\lambda)\, v, \quad \forall \lambda \in [0,1].$$

In fact we have

$$Au_\lambda(x) := -\frac{d}{dx} a \left(\frac{d}{dx} u_\lambda \right) = f,$$

which means that u_λ is the solution to (5) and moreover $u_0 = v$ is the minimal solution.

Let $\ell > 0$ be a real number. We denote by $\Omega_\ell = (-\ell, \ell) \times \Omega$. The points in $\mathbb{R}^{n+1}$ are denoted by (y, x) with $x \in \mathbb{R}^n$ and the gradient operator defined over $\mathbb{R}^{n+1}$ as

$$\nabla' = (\partial_y, \nabla) \text{ with } \nabla = (\partial_{x_1}, \partial_{x_2}, ..., \partial_{x_n}) .$$

We set

$$\mathcal{K}_\ell = \left\{ v \in W_0^{1,p}(\Omega_\ell) \mid v(y,.) \in \mathcal{K} \text{ a.e. in } (-\ell, \ell) \right\} .$$

This is a closed convex subset of $W_0^{1,p}(\Omega_\ell)$. For $f \in L^q(\Omega)$, let u_ℓ be the solution of the variational inequality

$$\left| \begin{array}{l} u_\ell \in \mathcal{K}_\ell, \\ \displaystyle\int_{\Omega_\ell} |\partial_y u_\ell|^{p-2} \partial_y u_\ell \partial_y (v - u_\ell)\, dxdy + \int_{-\ell}^{\ell} \langle A u_\ell, v - u_\ell \rangle dy \\ \displaystyle\geq \int_{\Omega_\ell} f(x)(v - u_\ell)\, dxdy, \ \forall v \in \mathcal{K}_\ell. \end{array} \right. \qquad (6)$$

It is clear that all the foregoing hypotheses assumed on the monotone operator A can be adapted to the operator

Lemma

Suppose that $f \in L^q(\Omega)$ is nonnegative and the assumptions (1)-(4) are satisfied. Then

(*i*) $(u_\ell)_{\ell>0}$ *is a nondecreasing sequence of nonnegative functions bounded above by any solution of Problem (5),*

(*ii*) *for all $\ell_0 > 0$, there exists a constant $C\left(\ell_0\right)$ independent of ℓ such that*

$$|u_\ell|_{1,p,\Omega_{\ell_0}} \leq C\left(\ell_0\right).$$

Lemma

Suppose that $f \in L^q(\Omega)$ is nonnegative and the assumptions (1)-(4) are satisfied. Then

- (i) $(u_\ell)_{\ell>0}$ *is a nondecreasing sequence of nonnegative functions bounded above by any solution of Problem (5),*
- (ii) *for all $\ell_0 > 0$, there exists a constant $C(\ell_0)$ independent of ℓ such that*
$$|u_\ell|_{1,p,\Omega_{\ell_0}} \leq C(\ell_0).$$

Lemma

Under the assumptions of above Lemma, the solution u_ℓ of (6) converges to $\tilde{u}$, as ℓ goes to $+\infty$, a solution of (5).

Theorem

Suppose that $f \in L^q(\Omega)$ is nonnegative and the assumptions (1)-(4) are satisfied. Then, there exists a minimal solution of (5) i.e.

$$\tilde{u}(x) = \min\{u(x),\ u \text{ solution to } (5)\}, \quad \tilde{u} \in \mathcal{K}$$

is solution to (5). Moreover, if u_1 and u_2 are the minimal solutions of (5) obtained by replacing f with f_1 and f_2 respectively, then, if $f_1 \leq f_2$, we have $u_1 \leq u_2$.

Remark

The results of the theorem remain true for a nonnegative distribution f in $W^{-1,q}(\Omega)$.

Employing the results of the previous section, we aim to extend the study to more general variational inequalities.

$$\left| \begin{array}{l} u \in \mathcal{K}, \\ \langle Au, v - u \rangle \geq \displaystyle\int_{\Omega} F(x, u)(v - u)\, dx, \quad \forall v \in \mathcal{K}, \end{array} \right. \tag{7}$$

where $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a nonnegative monotone Carathéodory function satisfying

$$\left| \begin{array}{l} F(x, \cdot) : \mathbb{R} \to \mathbb{R} \text{ is continuous and nondecreasing for a.e. } x \in \Omega, \\ F(\cdot, r) : \Omega \to \mathbb{R} \text{ is measurable } \forall r \in \mathbb{R}, \end{array} \right. \tag{8}$$

$$F(x, u) \in L^q(\Omega), \quad \forall u \in L^{p^*}(\Omega), \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}. \tag{9}$$

Here we will give a general condition related to our technique of construction but the existence of the minimal solution remains our main goal. Let us define the sequence of functions u_n as follows

$$\left|\begin{array}{l} u_0 = 0, \\ u_n \in \mathcal{K}, \\ \langle Au_n, v - u_n\rangle \geq \int_\Omega F\left(x, u_{n-1}\right)\left(v - u_n\right) dx, \quad \forall v \in \mathcal{K}, \end{array}\right. \tag{10}$$

where u_n is the minimal solution of the variational inequality in the last line of (10). Its existence is guaranteed by Theorem 4 since $F\left(x, u_{n-1}\right) \in L^q\left(\Omega\right)$.

We also denote

$$F_\infty := \lim_{n\to\infty} F(., u_n),$$

which may also be infinite on some subset. Assume that

$$F_\infty \in L^q(\Omega)\,. \tag{11}$$

Note that the above assumption is satisfied, for example, if $\sup_{r\geq 0} F(., r) \in L^q(\Omega)\,.$

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Note that the above assumption is satisfied, for example, if $\sup_{r\geq 0} F(., r) \in L^q(\Omega)$. Then the following lemma gives a characteristic property about the existence of a solution to (7) related to the above scheme.

Lemma

Let F be a nonnegative function satisfying the hypotheses (8), (9) and suppose that the assumptions (2)-(4) are fulfilled. If (11) is satisfied then u_∞, the limit of u_n, belongs to $\mathcal{K}$ and is a solution to (7).

The following theorem shows that (11) is more than just a simple condition and u_∞ is more than just a simple solution of (7).

Theorem

Under the assumptions (1)-(4), (8), (9), we have the equivalence between the following assertions

- **i)** *(7) has at least one solution,*
- **ii)** *(7) has a minimal solution,*
- **iii)** *the hypothesis (11) holds.*

Moreover if the hypothesis (11) holds, then u_∞, the limit of u_n, belongs to $\mathcal{K}$ and is the minimal solution to (7) i.e.

$$u_\infty(x) = \min\{u(x),\ u \text{ solution to } (7)\} \quad a.e. \ \ on\ \Omega. \tag{12}$$

A variant of the above result

Assume that $\bar{F}:\Omega\times\mathbb{R}^+\to\mathbb{R}$ is a Carathéodory function satisfying the following Lipschitz condition:

$$\left|\bar{F}(x,r)-\bar{F}(x,s)\right|\le k\left|r-s\right|,\quad x\in\Omega,\ \ r,s\in\mathbb{R}^+, \tag{13}$$

$$\bar{F}(x,0)\in L^q(\Omega)\ \text{ is a nonnegative function.} \tag{14}$$

Then, as a consequence of Theorem 6, we have

Corollary

Assume that the assumptions (1)-(4), (13) and (14) are satisfied and for $p\ge\frac{2n+2}{n+2}$ there exists a solution to

$$\left|\begin{array}{l} u\in\mathcal{K}, \\ \langle Au, v-u\rangle\ge\displaystyle\int_\Omega \bar{F}(x,u)(v-u)\,dx,\ \ \forall v\in\mathcal{K}. \end{array}\right. \tag{15}$$

Then (15) has a minimal solution.

Let ω be a bounded open subset of $\mathbb{R}^{n-1}$, $n \geq 2$, and $\mathcal{K}_\omega$ be a closed convex subset of $W_0^{1,p}(\omega)$ containing 0 and such that

$$\max(u,v),\ \min(u,v) \in \mathcal{K}_\omega,\ \ \forall u,v \in \mathcal{K}_\omega. \tag{16}$$

For $x \in \mathbb{R}\times\omega$ we set $x = (x_1, X_2)$ with $X_2 = (x_2, ..., x_n)$. Also, we denote by $\mathcal{K}$ the closed convex subset of $W_{loc}^{1,p}(\mathbb{R}\times\overline{\omega}) = \cup_{a>0} W^{1,p}((-a,a)\times\omega)$ defined by

$$\begin{aligned}\mathcal{K} &:= W_{loc}^{1,p}(\mathbb{R};\mathcal{K}_\omega)\\ &:= \Big\{ v \in W_{loc}^{1,p}(\mathbb{R}\times\overline{\omega}) \mid v = 0 \text{ on } \mathbb{R}\times\partial\omega \text{ and } v(x_1,.) \in \mathcal{K}_\omega \text{ for a.e. } x_1\end{aligned}$$

Then for a nonnegative f in $L^q_{loc}\left(\mathbb{R}, L^q\left(\omega\right)\right)$, we consider the following nonlinear variational inequality defined on the infinite cylinder $\Omega = \mathbb{R}\times\omega$

$$\left| \begin{array}{l} u \in \mathcal{K}, \\ \displaystyle\int_{\mathbb{R}\times\omega} a(x,u,\nabla u)\cdot\nabla\left(\left(v-u\right)\right)dx + \int_{\mathbb{R}\times\omega} a_0(x,u,\nabla u)\left(v-u\right)dx \\ \displaystyle\geq \int_{\mathbb{R}\times\omega} f\left(v-u\right)dx, \\ \qquad\qquad\qquad \forall v \in \mathcal{K},\ \forall\varphi\in\mathcal{D}\left(\mathbb{R}\right),\ \varphi\geq 0. \end{array}\right. \tag{17}$$

We also assume that there exists $h \in L^q\left(\omega\right)$ such that

$$f\left(x_1, X_2\right) \leq h\left(X_2\right) \text{ for a.e. } \left(x_1, X_2\right) \in \mathbb{R}\times\omega. \tag{18}$$

Then for a nonnegative f in $L^q_{loc}(\mathbb{R}, L^q(\omega))$, we consider the following nonlinear variational inequality defined on the infinite cylinder $\Omega = \mathbb{R}\times\omega$

$$\left| \begin{array}{l} u \in \mathcal{K}, \\ \displaystyle\int_{\mathbb{R}\times\omega} a(x,u,\nabla u)\cdot\nabla(\varphi(v-u))\,dx + \int_{\mathbb{R}\times\omega} a_0(x,u,\nabla u)\varphi(v-u)\,dx \\ \displaystyle\geq \int_{\mathbb{R}\times\omega} f\varphi(v-u)\,dx, \\ \qquad\qquad\qquad \forall v\in\mathcal{K},\ \forall\varphi\in\mathcal{D}(\mathbb{R}),\ \varphi\geq 0. \end{array} \right. \tag{17}$$

We also assume that there exists $h \in L^q(\omega)$ such that

$$f(x_1, X_2) \leq h(X_2) \text{ for a.e. } (x_1, X_2) \in \mathbb{R}\times\omega. \tag{18}$$

For $\ell > 0$, we set

$$\Omega_\ell = (-\ell, \ell)^2 \times \omega,$$

and for simplicity we also set $\langle .,. \rangle_{\ell,\omega}$ instead of $\int_{-\ell}^{\ell} \langle .,. \rangle_\omega \, dx_1$. We denote by (y, x_1, X_2) the points in Ω_ℓ and by $\mathcal{K}_\ell$ the closed convex subset of $W_0^{1,p}(\Omega_\ell)$ defined by

$$\mathcal{K}_\ell := \left\{ v \in W_0^{1,p}(\Omega_\ell) \mid v(y, x_1, .) \in \mathcal{K}_\omega, \text{ a.e. in } (-\ell, \ell)^2 \right\}.$$

Then consider u_ℓ solution to

$$\left| \begin{array}{l} u_\ell \in \mathcal{K}_\ell, \\ \displaystyle\int_{\Omega_\ell} |\partial_y u_\ell|^{p-2} \partial_y u_\ell \partial_y (v - u_\ell)\, dxdy + \int_{-\ell}^{\ell} \langle A u_\ell, v - u_\ell \rangle_{\ell,\omega} \, dy \\ \qquad\qquad \displaystyle\geq \int_{\Omega_\ell} f(x_1, X_2)(v - u_\ell)\, dxdy, \quad \forall v \in \mathcal{K}_\ell, \end{array} \right. \tag{19}$$

where A is our operator. Under the above assumptions, the problems (19) has a unique solution $u_\ell \in \mathcal{K}_\ell$. Then, we have

Theorem

The sequence $(u_\ell)_{\ell>0}$ is a nonnegative nondecreasing sequence in ℓ converging towards some $\tilde{u}$, as ℓ goes to $+\infty$, a nonnegative solution of (17).
There exists a minimal nonnegative solution to (17) i.e. $\tilde{u} = \min\{u \text{ solution to (17)}, u \geq 0\} \in \mathcal{K}$ is a solution to (17).
Moreover, let u_1 and u_2 be the minimum of nonnegative solutions to (17) obtained by replacing f with f_1 and f_2 respectively. Then if $f_1 \leq f_2$, we have $u_1 \leq u_2$.

The same result cane shown for the following nonlinear variational inequality

$$\left|\begin{array}{l} u \in \mathcal{K}, \\ \displaystyle\int_{\mathbb{R}\times\omega} a(x,u,\nabla u)\cdot\nabla\left(\varphi\left(v-u\right)\right)dx + \int_{\mathbb{R}\times\omega} a_0(x,u,\nabla u)\varphi\left(v-u\right)dx \\ \qquad\qquad \displaystyle\geq \int_{\mathbb{R}\times\omega} F\left(x,u\right)\varphi\left(v-u\right)dx, \;\; \forall v\in\mathcal{K}, \; \forall\varphi\in\mathcal{D}\left(\mathbb{R}\right), \end{array}\right. \tag{20}$$

where the function F is defined as and we assume that

$$h\left(X_2,r\right) := \sup_{x_1\in\mathbb{R}} F\left(x_1,X_2,r\right),$$

and

$$h\left(X_2,u\right) \in L^q\left(\omega\right), \;\; \forall u \in L^{p^*}\left(\omega\right). \tag{21}$$

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