Asymptotic analysis of some PDE's

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We consider, on $\Omega = (0,1) \times (0,1)$, a diffusion problem where the diffusion velocity is very small in the direction x_1 ($\varepsilon \simeq 0$)

$$\begin{cases} -\varepsilon^2 \partial^2_{x_1} u_\varepsilon - \partial^2_{x_2} u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

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When $\varepsilon \to 0$ the candidate limit is u_0 solution to

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Different issues are developed around the study of the limit

$$u_\varepsilon \to u_0 \ \text{ lorsque } \varepsilon \to 0.$$

$X_1 \in \mathbb{R}^p, X_2 \in \mathbb{R}^{n-p}$, Ω : a bounded open subset of $\mathbb{R}^n$.
$\Omega_{X_1} = \{ X_2 \mid (X_1, X_2) \in \Omega \}, \Omega_{X_2} = \{ X_1 \mid (X_1, X_2) \in \Omega \}$.

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Let $A = \big(a_{ij}(x)\big)$ be a $n \times n$ definite positive matrix. We decompose A into four blocks by writing

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11}, A_{22} are respectively $p \times p$ and $(n-p) \times (n-p)$ matrices.

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Let us consider the following elliptic problem

$$\begin{cases} \displaystyle\int_\Omega A_\varepsilon \nabla u_\varepsilon \cdot \nabla v \, dx = \langle f, v \rangle, \\ u_\varepsilon \in H_0^1(\Omega), \end{cases} \tag{1}$$

$f \in L^2(\Omega)$.

When $\varepsilon \to 0$ the candidate limit of u_ε is $u_0 = u_0(X_1, \cdot)$ defined for a.e. $X_1 \in \Pi_\Omega$ as solution to $v \in H_0^1(\Omega)$,

$$\begin{cases} \displaystyle\int_{\Omega_{X_1}} A_{22}\nabla_{X_2} u_0(X_1, X_2) \cdot \nabla_{X_2} v(X_2)\, dX_2 = \int_{\Omega_{X_1}} f(X_1, X_2) v(X_2)\, dX_2 \\ u_0(X_1, \cdot) \in H_0^1(\Omega_{X_1}). \end{cases} \tag{2}$$

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Theorem

We have

$$\varepsilon \nabla_{X_1} u_\varepsilon \longrightarrow 0, \qquad u_\varepsilon \longrightarrow u_0, \quad \nabla_{X_2} u_\varepsilon \longrightarrow \nabla_{X_2} u_0 \ \text{ in } L^2(\Omega).$$

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- $u_\varepsilon \rightharpoonup u_0$, $\nabla_{X_2} u_\varepsilon \rightharpoonup \nabla_{X_2} u_0$ and $\varepsilon \nabla_{X_1} u_\varepsilon \rightharpoonup 0$ in $L^2(\Omega)$.

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- Passing to the limit, we obtain

$$\int_\Omega A_{22} \nabla_{X_2} u_0 \cdot \nabla_{X_2} v \, dx = \int_\Omega f v \, dx \quad \forall\, v \in H_0^1(\Omega).$$

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$$I_\varepsilon = \int_\Omega A_\varepsilon \begin{pmatrix} \nabla_{X_1} u_\varepsilon \\ \nabla_{X_2}(u_\varepsilon - u_0) \end{pmatrix} \cdot \begin{pmatrix} \nabla_{X_1} u_\varepsilon \\ \nabla_{X_2}(u_\varepsilon - u_0) \end{pmatrix} dx \to 0. \qquad (3)$$

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$u_\varepsilon \longrightarrow u_0 \qquad \nabla_{X_2} u_\varepsilon \longrightarrow \nabla_{X_2} u_0$ in $L^2(\Omega)$.

$\Omega = \omega_1 \times \omega_2$ où $\omega_1 \subset \mathbb{R}^p$, $\omega_2 \subset \mathbb{R}^{n-p}$. $\Omega_{X_1} = \omega_2$, $\Omega_{X_2} = \omega_2$

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$$\begin{cases} \displaystyle\int_{\omega_2} A_{22}\left(X_1, X_2\right) \nabla_{X_2} u_0 \cdot \nabla_{X_2} v \, dX_2 = \int_{\omega_2} f\left(X_1, X_2\right) v \, dX_2 \\ u_0\left(X_1, \cdot\right) \in H_0^1(\omega_2). \end{cases} \tag{4}$$

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Note that

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Nevertheless, we can show that ($\omega_1' \subset\subset \omega_1$)

$$|u_\varepsilon - u_0|_{L^2(\omega_1' \times \omega_2)}, \ \left|\nabla_{X_2}(u_\varepsilon - u_0)\right|_{L^2(\omega_1' \times \omega_2)} = O(\varepsilon),$$

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Moreover if we suppose that $A_{12} = A_{21}^T = 0$, Then we have

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$$\left|\nabla_{X_1}(u_\varepsilon - u_0)\right|_{L^2(\omega_1' \times \omega_2)} = o(1), \quad u_\varepsilon \to u_0 \text{ in } H^1(\Omega').$$

Cylindrical domains

A necessary and sufficient condition to obtain the above convergence

$$\int_\Omega A_{12}\nabla_{X_2}u_0\cdot\nabla_{X_1}v\,dx+\int_\Omega A_{21}\nabla_{X_1}u_0\cdot\nabla_{X_2}v\,dx=0\quad\forall v\in H_0^1(\Omega). \tag{6}$$

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For example if $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ we have

$$u_\varepsilon \to u_0 \text{ in } H^1\left(\Omega'\right) \iff u_0 \text{ is independent of } x_1 .$$

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If $u_0 = u_0(X_2)$, the hypothesis (7) can be reduced to

$$\int_\Omega A_{12}\nabla_{X_2} u_0 \cdot \nabla_{X_1} v\, dx = 0 \ \ \forall v \in H_0^1(\Omega).$$

Moreover

$$|u_\varepsilon - u_0|_{H^1(\Omega')} \le Ce^{-\frac{\alpha}{\varepsilon}}.$$

A rate of convergence as

$$u_\varepsilon - u_0 = o\left(\varepsilon\right) \;\text{ in } L^2\left(\omega_1' \times \omega_2\right), \quad \omega_1' \subset\subset \omega_1$$

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In order to improve the rate of convergence, we consider an approximation w_ε of u_ε depending on ε expressed as a polynomial in ε i.e.

$$w_\varepsilon^N = u_0 + \varepsilon u_1 + \cdots + \varepsilon^N u_N.$$

Formally, if we substitute the asymptotic expansion into (1), we then deduce that the coefficient u_N are solutions, for a.e. $X_1 \in \omega_1$, to

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and for $N \geq 1$

$$\begin{cases} -\nabla_{X_2} \cdot \left(A_{22} \nabla_{X_2} u_N(X_1, \cdot)\right) = & \nabla_{X_1} \cdot (A_{11} \nabla_{X_1} u_{N-2}(X_1, \cdot)) \\ & + \nabla_{X_1} \cdot \left(A_{12} \nabla_{X_2} u_{N-1}(X_1, \cdot)\right) \\ & + \nabla_{X_2} \cdot \left(A_{21} \nabla_{X_1} u_{N-1}(X_1, \cdot)\right) \quad \text{in } \omega_2, \\ u_N(X_1, \cdot) \in H_0^1(\omega_2). & \end{cases}$$

$(u_{-1} = 0)$

In order to define w_ε as function we need the following regularity theorem.

Theorem

Let $m \in \mathbb{N}$ and $g \in L^2(\Omega)$ such that

$$D^m_{X_1} g \in L^2(\Omega), \; D^m_{X_1} A_{22} \in L^\infty(\Omega).$$

The elliptic boundary problem

$$\begin{cases} -\nabla_{X_2} \cdot \left(A_{22} \nabla_{X_2} u(X_1, \cdot)\right) = g(X_1, \cdot) & \text{in } \omega_2, \\ u(X_1, \cdot) \in H^1_0(\omega_2). \end{cases} \tag{8}$$

has a unique solution u satisfying

$$D^m_{X_1} u, \; D^m_{X_1} \left(\nabla_{X_2} u\right) \in L^2(\Omega). \tag{9}$$

($D^m_{X_i}$ denotes the mixed derivatives in X_i of order up to m).

To ensure the existence of u_N in $H^1(\Omega)$ we need

$$D^2_{X_1} A_{11},\ D^2_{X_1} A_{12},\ D^1_{X_1} A_{22} \in L^\infty(\Omega),$$
$$D^3_{X_1} u_{N-2},\ D^2_{X_1} u_{N-1},\ D^2_{X_1} D^1_{X_2} u_{N-1} \in L^2(\Omega).$$

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We set $R_N(\cdot;\varepsilon) = u_\varepsilon - \sum_{i=0}^N \varepsilon^i u_i$.

Theorem

For any $\omega_1' \subset\subset \omega_1$, it holds that, when $\varepsilon \to 0$, in $L^2\left(\omega_1' \times \omega_2\right)$,

$$R_N(\cdot;\varepsilon),\ \nabla_{X_2} R_N(\cdot;\varepsilon) = O\left(\varepsilon^{N+1}\right),\ \ \nabla_{X_1} R_N(\cdot;\varepsilon) = O\left(\varepsilon^N\right). \tag{11}$$

This implies that

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The tool:

$$\text{supp}\rho \subset \omega_1,\ \ \rho = 1 \text{ on } \omega_1',\ \ 0 \le \varrho \le 1 \ \text{ et } \ \left|\nabla_{X_1} \varrho\right| \le C. \tag{12}$$

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We set $R_N(\cdot;\varepsilon)=u_\varepsilon-\sum_{i=0}^N\varepsilon^i u_i$.

Theorem

For any $\omega_1'\subset\subset\omega_1$, it holds that, when $\varepsilon\to 0$, in $L^2\left(\omega_1'\times\omega_2\right)$,

$$R_N(\cdot;\varepsilon),\ \nabla_{X_2}R_N(\cdot;\varepsilon)=O\left(\varepsilon^{N+1}\right),\ \ \nabla_{X_1}R_N(\cdot;\varepsilon)=O\left(\varepsilon^N\right). \tag{11}$$

The tool:

$$\text{supp}\rho\subset\omega_1,\ \rho=1 \text{ on } \omega_1',\ \ 0\le\varrho\le 1\ \text{ et }\ \left|\nabla_{X_1}\varrho\right|\le C. \tag{12}$$

Then testing with $v=\rho^2R_\varepsilon\in H_0^1(\Omega)$, Suppose that (10) holds for $N+1$,

$$D_{X_1}^{N+1} A_{11},\ D_{X_1}^{N+1+1} A_{12},\ D_{X_1}^{N+1} A_{21},\ D_{X_1}^{N+1+1} A_{22} \in L^\infty(\Omega),$$
$$D_{X_1}^{N+1+1} f \in L^2(\Omega).$$

Suppose that (10) holds for $N+1$, then in particular we have

$$\left|\nabla_{X_1} R_{N+1}(\cdot;\varepsilon)\right|_{L^2(\omega_1'\times\omega_2)} = O\left(\varepsilon^{N+1}\right). \tag{13}$$

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This implies

$$\nabla_{X_1} R_N(\cdot;\varepsilon) = \nabla_{X_1} R_{N+1}(\cdot;\varepsilon) + \varepsilon^{N+1}\nabla_{X_1} u_{N+1} = O\left(\varepsilon^{N+1}\right)$$

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Then we end up with

$$|R_N(\cdot;\varepsilon)|_{H^1(\omega_1'\times\omega_2)} = O(\varepsilon^{N+1}). \tag{14}$$

Theorem

Consider the following assertions, for any $\omega_1' \subset\subset \omega_1$,

Theorem

Consider the following assertions, for any $\omega_1' \subset\subset \omega_1$,
i) $u_{N+1} = 0$,,
ii) *the following condition holds,*

$$\nabla_{X_1} \cdot (A_{11} \nabla_{X_1} u_{N-1}) + \nabla_{X_1} \cdot (A_{12} \nabla_{X_2} u_N) + \nabla_{X_2} \cdot (A_{21} \nabla_{X_1} u_N) = 0 \text{ in } \Omega, \tag{15}$$

iii) *as* $\varepsilon \longrightarrow 0$

$$\frac{1}{\varepsilon^{N+1}} \left[u_\varepsilon - (u_0 + \varepsilon u_1 + \cdots + \varepsilon^N u_N) \right] \rightharpoonup 0 \text{ in } L^2(\omega_1' \times \omega_2),$$

iv) *as* $\varepsilon \longrightarrow 0$,

$$u_\varepsilon - (u_0 + \varepsilon u_1 + \cdots + \varepsilon^N u_N) = O(\varepsilon^{N+2}) \quad \text{in } \mathcal{V}(\omega_1' \times \omega_2),$$

Theorem

v) $u_{N+1} = 0, \quad u_{N+2} = 0,$
vi) *the conditions (**??**) and the following hold*

$$\nabla_{X_1} \cdot \left(A_{11} \nabla_{X_1} u_N\right) = 0 \quad \text{in } \mathcal{D}'(\Omega)\,.$$

vii) $u_k = 0, \quad \forall k > N,$
viii) *as* $\varepsilon \longrightarrow 0,$

$$\frac{1}{\varepsilon^{N+2}} \nabla_{X_2} \left[u_\varepsilon - (u_0 + \varepsilon u_1 + \cdots + \varepsilon^N u_N) \right] \rightharpoonup 0 \quad \text{in } L^2\left(\omega_1' \times \omega_2\right)$$

ix) *as* $\varepsilon \longrightarrow 0$ *and for some* $\eta > 0,$

$$u_\varepsilon - (u_0 + \varepsilon u_1 + \cdots + \varepsilon^N u_N) = O\left(\exp(\tfrac{-\eta}{\varepsilon})\right) \quad \text{in } H^1(\omega_1' \times \omega_2).$$

Theorem

Then we have

$$i) \Leftrightarrow ii) \Leftrightarrow iii \Leftrightarrow iv) \Leftarrow v) \Leftrightarrow vi) \Leftrightarrow vii) \Leftrightarrow viii) \Leftrightarrow ix).$$

If A is a diagonal matrix by block

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix},$$

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and for $N \geq 1$

$$-\nabla_{X_2} \cdot (A_{22} \nabla_{X_2} u_N(X_1, \cdot)) = \nabla_{X_1} \cdot (A_{11} \nabla_{X_1} u_{N-2}(X_1, \cdot)) \quad \text{in } \omega_2.$$

If A is a diagonal matrix by block

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we have

$$u_{2k+1} = 0, \qquad k \in \mathbb{N}.$$

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we have

$$u_{2k+1} = 0, \qquad k \in \mathbb{N}.$$

Then, it follows that

$$\left| R_{2N}(\cdot;\varepsilon) \right|_{L^2(\omega_1' \times \omega_2)}, \qquad \left| \nabla_{X_2} R_{2N}(\cdot;\varepsilon) \right|_{L^2(\omega_1' \times \omega_2)} = O(\varepsilon^{2N+2}),$$
$$\left| \nabla_{X_1} R_{2N}(\cdot;\varepsilon) \right|_{L^2(\omega_1' \times \omega_2)} = O(\varepsilon^{2N+1}).$$

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$$|R_{2N}(\cdot;\varepsilon)|_{L^2(\omega_1' \times \omega_2)}, \qquad \left|\nabla_{X_2} R_{2N}(\cdot;\varepsilon)\right|_{L^2(\omega_1' \times \omega_2)} = O(\varepsilon^{2N+2}),$$
$$\left|\nabla_{X_1} R_{2N}(\cdot;\varepsilon)\right|_{L^2(\omega_1' \times \omega_2)} = O(\varepsilon^{2N+1}).$$

Moreover, if we replace in (10), $2N$ by $2N+2$

$$|R_{2N}(\cdot;\varepsilon)|_{H^1(\omega_1' \times \omega_2)} = O(\varepsilon^{2N+2}). \tag{15}$$

We suppose that

$$A = \begin{pmatrix} A_{11}\left(X_1, X_2\right) & A_{12}\left(X_2\right) \\ A_{12}\left(X_2\right) & A_{22}\left(X_2\right) \end{pmatrix}$$

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$$A = \begin{pmatrix} A_{11}(X_1, X_2) & A_{12}(X_2) \\ A_{12}(X_2) & A_{22}(X_2) \end{pmatrix}$$

where A_{11} is a first order polynomial in X_1 and f is a polynomial of degree $k \in \mathbb{N}$, i.e.

$$f(X_1, X_2) = \sum_{|\alpha| \leq k} X_1^{\alpha} f_{\alpha}(X_2), \qquad f_{\alpha} \in L^2(\omega_2).$$

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$$f(X_1, X_2) = \sum_{|\alpha| \leq k} X_1^{\alpha} f_{\alpha}(X_2), \qquad f_{\alpha} \in L^2(\omega_2).$$

Under the above assumptions, there exist constants $C, \alpha > 0$ independents of $\varepsilon > 0$ such that

$$|R_{2k}(\cdot;\varepsilon)|_{H^1(\omega_1' \times \omega_2)} \leq Ce^{\frac{-\alpha}{\varepsilon}}. \tag{16}$$

Formally, if we substitute the asymptotic expansion into (1), we then deduce that the coefficient u_N are solutions, for a.e. $X_1 \in \omega_1$, to

$$\begin{cases} -\nabla_{X_2} \cdot \left(A_{22} \nabla_{X_2} u_0(X_1, \cdot)\right) = f(X_1, \cdot) \quad \text{in } \omega_2, \\ u_0(X_1, \cdot) \in H_0^1(\omega_2), \end{cases}$$

and for $N \geq 1$

$$\begin{cases} -\nabla_{X_2} \cdot \left(A_{22} \nabla_{X_2} u_N(X_1, \cdot)\right) = & \nabla_{X_1} \cdot (A_{11} \nabla_{X_1} u_{N-2}(X_1, \cdot)) \\ & +\nabla_{X_1} \cdot \left(A_{12} \nabla_{X_2} u_{N-1}(X_1, \cdot)\right) \\ & +\nabla_{X_2} \cdot \left(A_{21} \nabla_{X_1} u_{N-1}(X_1, \cdot)\right) \quad \text{in } \omega_2, \\ u_N(X_1, \cdot) \in H_0^1(\omega_2). \end{cases}$$

$(u_{-1} = 0)$

We consider the problem

$$\begin{cases} -\varepsilon^2 \partial^2_{X_1} u_\varepsilon - \Delta_{X_2} u_\varepsilon = f \quad \text{in } \tilde{\Omega} = (0,1) \times \omega, \\ u_\varepsilon = 0 \ \text{ on } \partial\tilde{\Omega} \setminus \{0\} \times \omega, \ \ \dfrac{\partial u_\varepsilon}{\partial X_1} = 0 \ \text{ on } \{0\} \times \omega. \end{cases} \tag{17}$$

We have

$$u_\varepsilon \to u_0 \qquad \text{in } H^1((0, 1-\alpha) \times \omega)$$

The aim next is to describe the behaviour of u_ε near the section $\{1\} \times \omega$:

$$u_\varepsilon - u_0 - w_\varepsilon \to 0 \qquad \text{in } H^1_0(\Omega). \qquad (w_\varepsilon \text{ is a corrector })$$

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It is clear that w_ε has to satisfy

$$w_\varepsilon \to 0 \qquad \text{in } \ H^1_0((0, 1-\alpha) \times \omega) \text{ et } w_\varepsilon = -u_0 \ \text{ on } \ \{1\} \times \omega.$$

The choice of the corrector is based on:

Lemma

Si w is a solution to

$$\begin{cases} \Delta w = 0 \ \ in \ (0, +\infty) \times \omega, \\ w = -u_0 \ \ on \ \ \{0\} \times \omega, \quad w = 0 \ \ on \ \ (0, +\infty) \times \partial\omega, \end{cases}$$

There exists $C > 0$, $\alpha > 0$ independent of ε such that

$$\int_{S_{\frac{1}{\varepsilon}}} |\nabla w|^2 \, dx \leq C e^{-\frac{\alpha}{\varepsilon}} \int_{S_0} |\nabla w|^2 \, dx, \quad S_\ell = (\ell, +\infty) \times \omega.$$

The corrector is defined by

$$w_\varepsilon (X_1, X_2) = w \left(\frac{1 - X_1}{\varepsilon}, X_2 \right).$$

By consequence, we have

Theorem

If $f \in L^2\left(\tilde{\Omega}\right)$, $\partial_{X_1} f \in L^2\left(\tilde{\Omega}\right)$, *then*

$$\left|u_\varepsilon - u_0 - w_\varepsilon\right|_{L^2(\tilde{\Omega})}, \ \left|\nabla_{X_2}\left(u_\varepsilon - u_0 - w_\varepsilon\right)\right|_{L^2(\tilde{\Omega})} = o(\varepsilon),$$
$$\left|\partial_{X_1}\left(u_\varepsilon - u_0 - w_\varepsilon\right)\right|_{L^2(\tilde{\Omega})} = o(1).$$

Moreover, if u_0 *is independent of* X_1 *we obtain*

$$\int_{\tilde{\Omega}} \left|\nabla\left(u_\varepsilon - u_0 - w_\varepsilon\right)\right|^2 dx \leq Ce^{-\frac{\alpha}{\varepsilon}}.$$

Abstract problems

$$-\varepsilon\partial_{x_1}^2 u_\varepsilon - \partial_{x_2}^2 u_\varepsilon = f.$$

$$-\varepsilon\partial^2_{x_1}u_\varepsilon - \partial^2_{x_2}u_\varepsilon = f.$$

An abstract approach to this theory is given in (E. Sanchez-Palencia 1992, S. Zhang 2006) where the following operator equation is considered

$$\varepsilon A u_\varepsilon + B u_\varepsilon = f.$$

where A and B are linear operators defined on Hilbert spaces.

There are also some previous works on singular perturbations of variational inequalities (Lions, Stampacchia 1968, Stampacchia 1969)

$$\begin{cases} \varepsilon \langle Au_\varepsilon, v - u_\varepsilon \rangle_V + \langle Bu_\varepsilon, v - u_\varepsilon \rangle_W \geq \langle f, v - u_\varepsilon \rangle_{V \cap W}, \ \forall v \in K, \\ u_\varepsilon \in K, \end{cases}$$

$A : V \to V'$, $B : W \to W'$: monotone operators $V \subset W$.

$$\begin{cases} \varepsilon \langle Au_\varepsilon, v - u_\varepsilon \rangle_V + \langle Bu_\varepsilon, v - u_\varepsilon \rangle_W \geq \langle f, v - u_\varepsilon \rangle_{V \cap W}, \ \forall v \in K, \\ u_\varepsilon \in K, \end{cases}$$

$A : V \to V'$, $B : W \to W'$: monotone, bounded, coercive and hemicontinuous.
$(V, |\cdot|_V)$ et $(W, |\cdot|_W)$: (separable, reflexive Banach spaces), $\overline{V \cap W} = V, W$.
$K \neq \varnothing$: a closed convex set of $V \cap W$:
$f \in (V \cap W)'$, $\varepsilon > 0$, there exists a solution $u_\varepsilon \in K$.

Theorem

Suppose that $f \in W'$ *and let* u_ε *be solution to* (**??**). *Then we have when* $\varepsilon \to 0$

$$(i)\ u_\varepsilon \text{ is bounded in } W \text{ independently of } \varepsilon, \tag{18}$$

$$(ii)\ \varepsilon u_\varepsilon \to 0 \text{ in } V, \tag{19}$$

$$(iii)\ \varepsilon A u_\varepsilon \to 0 \text{ in } V', \tag{20}$$

$$(iv)\ \langle \varepsilon A u_\varepsilon, u_\varepsilon \rangle_V \to 0. \tag{21}$$

u_ε is bounded in W,
Suppose that $|u_\varepsilon - v_0|_W$ is unbounded. For some sequence $\varepsilon_k \to 0$ one has then

$$|u_{\varepsilon_k} - v_0|_W \to +\infty.$$

Taking $v = v_0$ in the variational equation, we derive

$$\begin{aligned} \varepsilon_k \langle Au_{\varepsilon_k}, u_{\varepsilon_k} - v_0\rangle_V + \langle Bu_{\varepsilon_k}, u_{\varepsilon_k} - v_0\rangle_W &\leq \langle f, u_{\varepsilon_k} - v_0\rangle_W \\ &\leq |f|_{W'} \, |u_{\varepsilon_k} - v_0|_W \,. \end{aligned}$$

It follows that

$$\frac{\varepsilon_k \langle Au_{\varepsilon_k}, u_{\varepsilon_k} - v_0\rangle_V}{|u_{\varepsilon_k} - v_0|_W} + \frac{\langle Bu_{\varepsilon_k}, u_{\varepsilon_k} - v_0\rangle_W}{|u_{\varepsilon_k} - v_0|_W} \leq |f|_{W'} \,. \qquad (22)$$

This is impossible.

$\varepsilon u_\varepsilon \to 0$ in V,

Since

$$\varepsilon \left\langle Au_\varepsilon, u_\varepsilon - v_0 \right\rangle_V \leq C \tag{23}$$

for some constant C independent of ε. If $(u_\varepsilon - v_0)$ is not bounded in V, we have -up to a subsequence-

$$\varepsilon \left| u_\varepsilon - v_0 \right|_V \leq C \frac{\left| u_\varepsilon - v_0 \right|_V}{\left\langle Au_\varepsilon, u_\varepsilon - v_0 \right\rangle_V} \to 0$$

$\left\langle \varepsilon Au_\varepsilon, u_\varepsilon \right\rangle_V \to 0.$

From the monotonicity of A we have

$$\varepsilon \left\langle Au_\varepsilon, v \right\rangle_V \leq \varepsilon \left\langle Au_\varepsilon, u_\varepsilon \right\rangle_V + \left\langle Av, \varepsilon \left(v - u_\varepsilon \right) \right\rangle_V . \tag{24}$$

and from the variational inequality we get

$$\varepsilon \left\langle Au_\varepsilon, u_\varepsilon \right\rangle_V \leq \left\langle \varepsilon Au_\varepsilon, v_0 \right\rangle_V + C.$$

Thus,

$$\varepsilon \langle Au_\varepsilon, v - v_0 \rangle_V \leq C + \langle Av, \varepsilon (v - u_\varepsilon) \rangle_V , \tag{25}$$

Choosing $v \in v_0 + \mathcal{B}_1$, where $\mathcal{B}_1$ is the unit ball of V, we arrive to

$$\varepsilon \langle Au_\varepsilon, v_1 \rangle_V \leq C', \quad \forall v_1 \in \mathcal{B}_1,$$

and for some subsequence

$$\varepsilon Au_\varepsilon \rightharpoonup \psi \text{ in } V'.$$

Passing to the limit in (25) we derive

$$\langle \psi, v - v_0 \rangle_V \leq C, \quad \forall v \in V$$

and thus $\psi = 0$.

For any $v \in K$ we have by the variational inequality and the monotonicity of B

$$\varepsilon \left\langle Au_\varepsilon, u_\varepsilon \right\rangle_V \leq \left\langle \varepsilon Au_\varepsilon, v \right\rangle_V + \left\langle f, u_\varepsilon - v \right\rangle_W + \left\langle Bv, v - u_\varepsilon \right\rangle_W .$$

Let $(\varepsilon_k)_k$ be a sequence such that

$$\varepsilon_k \left\langle Au_{\varepsilon_k}, u_{\varepsilon_k} \right\rangle_V \to \limsup_{\varepsilon \to 0} \varepsilon \left\langle Au_\varepsilon, u_\varepsilon \right\rangle_V , \; u_{\varepsilon_k} \rightharpoonup \tilde{u} \text{ in } W.$$

Then passing to the limit

$$\limsup_{\varepsilon \to 0} \varepsilon \left\langle Au_\varepsilon, u_\varepsilon \right\rangle_V \leq \left\langle f, \tilde{u} - v \right\rangle_W + \left\langle Bv, v - \tilde{u} \right\rangle_W , \forall v \in K. \tag{26}$$

Since K is convex, there exists a sequence $v_n \in K$ such that $v_n \to \tilde{u}$. Thus

$$\limsup_{\varepsilon \to 0} \varepsilon \langle Au_\varepsilon, u_\varepsilon \rangle_V \leq 0.$$

Thanks to the monotonicity

$$\liminf_{\varepsilon \to 0} \varepsilon \langle Au_\varepsilon, u_\varepsilon \rangle_V \geq 0.$$

$\varepsilon Au_\varepsilon \rightharpoonup 0$ in V'.

For every $v_1 \in \mathcal{B}_1$,

$$\begin{aligned} \varepsilon \langle Au_\varepsilon, v_1 \rangle_V &\leq \varepsilon \langle Au_\varepsilon, u_\varepsilon \rangle_V + |Av_1|_{V'} \left(\varepsilon + |\varepsilon u_\varepsilon|_V\right) \\ &\leq \varepsilon \langle Au_\varepsilon, u_\varepsilon \rangle_V + C \left(\varepsilon + |\varepsilon u_\varepsilon|_V\right) \to 0. \end{aligned}$$

When $\varepsilon \to 0$, if $u_{\varepsilon_k} \rightharpoonup \tilde{u}$ in W, then u is a solution of

$$\begin{cases} \langle B\tilde{u}, v - \tilde{u}\rangle_W \geq \langle f, v - \tilde{u}\rangle_W , \quad \forall v \in \bar{K}^W, \\ \tilde{u} \in \bar{K}^W. \end{cases} \tag{27}$$

and $Bu_{\varepsilon_k} \rightharpoonup B\tilde{u}$ in W', $\langle Bu_{\varepsilon_k}, u_{\varepsilon_k}\rangle_W \to \langle B\tilde{u}, \tilde{u}\rangle_W$.

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$$\begin{cases} \langle B\tilde{u}, v - \tilde{u} \rangle_W \geq \langle f, v - \tilde{u} \rangle_W \,, \quad \forall v \in \bar{K}^W, \\ \tilde{u} \in \bar{K}^W. \end{cases} \tag{27}$$

and $Bu_{\varepsilon_k} \rightharpoonup B\tilde{u}$ in W', $\langle Bu_{\varepsilon_k}, u_{\varepsilon_k} \rangle_W \to \langle B\tilde{u}, \tilde{u} \rangle_W$. If B is strongly monotone in the sense that for some $\delta > 0$ and $\beta > 1$

$$\langle Bu - Bv, u - v \rangle_W \geq \delta \, |u - v|_W^\beta \,, \quad \forall v, u \in W \tag{28}$$

then the solution $\tilde{u}$ of (27) is unique and one has

$$u_\varepsilon \to \tilde{u} \ \text{ in } W.$$

If the variational inequality (27) has a solution in V, then u_ε is bounded in V and there exists always a sequence u_{ε_k} such that

$$u_{\varepsilon_k} \rightharpoonup \tilde{u} \text{ in } V \text{ and } W, \tag{29}$$

where $\tilde{u} \in K$ is solution to (27).
In addition if B satisfies (28), one has

$$|u_\varepsilon - \tilde{u}|_W = o\left(\varepsilon^{1/\beta}\right). \tag{30}$$

Let us consider the following variational inequality defined as

$$\begin{cases} \int_\Omega a(x, \nabla^\varepsilon u_\varepsilon) \cdot \nabla^\varepsilon (v_\varepsilon - u_\varepsilon)\, dx \geq \langle f, v_\varepsilon - u_\varepsilon \rangle_{W_0^{1,p}(\Omega)}, \qquad \forall v_\varepsilon \in K_\varepsilon, \\ u_\varepsilon \in K_\varepsilon \end{cases} \tag{31}$$

where $K_\varepsilon \neq \varnothing$ is a closed convex subset of $W_0^{1,p}(\Omega)$ for all $\varepsilon > 0$. We make the following standard assumptions: Carathéodory condition, Growth condition, Monotonicity, Coercivity.

Theorem

Assume in addition that there exists a sequence $(w_\varepsilon) \subset W_0^{1,p}(\Omega)$, $w_\varepsilon \in K_\varepsilon$ for all $\varepsilon > 0$, s.t.

$$\varepsilon \nabla_{X_1} w_\varepsilon \quad \textit{and} \quad \nabla_{X_2} w_\varepsilon \quad \textit{are bounded in } L^p(\Omega) \tag{32}$$

independently of ε, then u_ε satisfies the same estimates.

$$\begin{cases} \varepsilon \langle Au_\varepsilon, v - u_\varepsilon \rangle_V + \langle Bu_\varepsilon, v - u_\varepsilon \rangle_W \geq \langle f, v - u_\varepsilon \rangle_{V \cap W}, \ \forall v \in K, \\ u_\varepsilon \in K, \end{cases}$$

Examples:

Isotopic S.P.

$$\begin{cases} u_\varepsilon \in K_0 = \left\{ v \in H_0^1(\Omega) \,|\, v(x) \geq 0, \ a.e.\ x \in \Omega \right\}, \\ \varepsilon \int_\Omega a(\nabla u_\varepsilon) \cdot \nabla (v - u_\varepsilon)\, dx + \int_\Omega u_\varepsilon (v - u_\varepsilon)\, dx \geq \int_\Omega f(v - u_\varepsilon)\, dx, \end{cases}$$

$$\begin{cases} \varepsilon \langle Au_\varepsilon, v - u_\varepsilon \rangle_V + \langle Bu_\varepsilon, v - u_\varepsilon \rangle_W \geq \langle f, v - u_\varepsilon \rangle_{V \cap W}, \ \forall v \in K, \\ u_\varepsilon \in K, \end{cases}$$

Examples:

Anisotropic S.P.

$$\begin{cases} -\varepsilon \Delta_{X_1} u_\varepsilon - \Delta_{X_2} u_\varepsilon + g(x, u_\varepsilon) = f \quad \text{dans } \Omega, \\ u_\varepsilon \in H_0^1(\Omega) \cap L^p(\Omega), \end{cases}$$

$$\begin{cases} \varepsilon \left\langle Au_\varepsilon, v - u_\varepsilon \right\rangle_V + \left\langle Bu_\varepsilon, v - u_\varepsilon \right\rangle_W \geq \left\langle f, v - u_\varepsilon \right\rangle_{V\cap W}, \ \forall v \in K, \\ u_\varepsilon \in K, \end{cases}$$

Examples:
Anisotropic S.P.

$$\begin{cases} -\varepsilon \Delta_{p_1, X_1} u_\varepsilon - \Delta_{p_2, X_2} u_\varepsilon = f & \text{dans } \Omega, \\ u_\varepsilon = 0 & \text{sur } \partial\Omega. \end{cases}$$

The sets $\underline{\lim}_{\varepsilon\to 0} K_\varepsilon$ and $\overline{\lim}_{\varepsilon\to 0} K_\varepsilon$ are defined as

$$\begin{aligned} w &\in as - \underline{\lim}_{\varepsilon\to 0} K_\varepsilon \quad \text{iff} \\ \exists w_\varepsilon \in K_\varepsilon \;, \varepsilon \nabla_{X_1} w_\varepsilon ,\; \nabla_{X_2}(w_\varepsilon - w) &\to 0 \quad in\; L^p(\Omega) \,,\; \text{as } \varepsilon \to 0. \end{aligned}$$

$$\begin{aligned} w &\in aw - \overline{\lim}_{\varepsilon\to 0} K_\varepsilon \text{ iff} \\ \exists w_{\varepsilon_k} \in K_{\varepsilon_k} \;, \varepsilon_k \nabla_{X_1} w_{\varepsilon_k} ,\; \nabla_{X_2}(w_{\varepsilon_k} - w) &\rightharpoonup 0 \quad in\; L^p(\Omega) \,, \quad \text{as } \varepsilon_k \to 0. \end{aligned}$$

We say that the sequence (K_ε) of subsets of $W_0^{1,p}(\Omega)$ converges to $\mathcal{K}$ ($K_\varepsilon \overset{a}{\to} \mathcal{K}$), if

$$aw - \overline{\lim}_{\varepsilon\to 0} K_\varepsilon = as - \underline{\lim}_{\varepsilon\to 0} K_\varepsilon = \mathcal{K}.$$

where $\mathcal{K}$ is a closed convex set in

$$\mathcal{W}(\Omega) := \left\{ u \in L^p(\Omega) \,\middle|\, \nabla_{X_2} u \in [L^p(\Omega)]^{n-q} ,\;\; u(X_1, \cdot) \in W_0^{1,p}(\Omega_{X_1}) \,, \right.$$

Theorem

Assume that $K_\varepsilon \xrightarrow{a} \mathcal{K}$ *as* $\varepsilon \to 0$, *then -up to a subsequence- we have*

$$u_\varepsilon \rightharpoonup \tilde{u} \quad \varepsilon \nabla_{X_1} u_\varepsilon \rightharpoonup 0, \quad \nabla_{X_2} u_\varepsilon \rightharpoonup \nabla_{X_2} \tilde{u} \quad \text{in } L^p(\Omega), \tag{33}$$

where $\tilde{u}$ *is a solution to*

$$\begin{cases} \displaystyle\int_\Omega a(x, \nabla_{X_2} u) \cdot \nabla_{X_2}(v - u)\, dx \geq \langle f, v - u \rangle_{\mathcal{W}(\Omega)}, \quad \forall v \in \mathcal{K}, \\ u \in \mathcal{K} \end{cases}$$

Moreover, if a is strongly monotone then the previous convergences hold strongly.

Obstacle problems

For $\varepsilon > 0$, set

$$K_\varepsilon = \left\{ v \in W_0^{1,p}(\Omega) \mid v \geq \psi_\varepsilon \text{ a.e. in } \Omega \right\}$$

$\psi_\varepsilon \in W_0^{1,p}(\Omega)$. Assume that

$$\varepsilon \nabla_{X_1} \psi_\varepsilon \to 0, \quad \nabla_{X_2} \psi_\varepsilon \to \nabla_{X_2} \psi_0 \quad \text{in } L^p(\Omega) \text{ as } \varepsilon \to 0.$$

Then

$$K_\varepsilon \xrightarrow{a} \mathcal{K} := \{ v \in \mathcal{W}(\Omega) \mid v \geq \psi_0 \text{ a.e. in } \Omega \}.$$

An elasto-plastic problem

We set

$$K_\varepsilon = K_{\beta_\varepsilon} := \left\{ v \in W_0^{1,p}(\Omega) \mid \ |\nabla^\varepsilon v| \leq \beta_\varepsilon \text{ a.e. in } \Omega \right\},$$

$\beta_\varepsilon \in L^\infty(\Omega)$. Assume that

$$\varepsilon \nabla_{X_1} \beta_\varepsilon \to 0, \quad \nabla_{X_2} \beta_\varepsilon \to \nabla_{X_2} \beta_0 \quad \text{in } L^\infty(\Omega) \text{ as } \varepsilon \to 0.$$

where $\beta_0 > 0$ a.e. in Ω, then

$$K_{\beta_\varepsilon} \xrightarrow{a} \mathcal{K}_{\beta_0} := \left\{ v \in \mathcal{W}(\Omega) \mid \left| \nabla_{X_2} v \right| \leq \beta_0 \text{ a.e. in } \Omega \right\}.$$

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